Central Bank Corporate Bond Purchase Programs: Commitment Matters

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First Version: March 29, 2023
This Version: April 14, 2025

Over the past decade, the European Central Bank (ECB) and the Federal Reserve expanded the limits of unconventional monetary policy to directly provide firms with financing through corporate bond purchases. Empirical research has found that these programs led to increased leverage for directly targeted firms, as well as relatively higher payouts to shareholders but no relative increase in investment, contrary to the central banks' stated objectives. This paper makes the novel observation that both the ECB and Fed engaged in de facto unsecured debt intervention in financially unconstrained firms. I show that the stated stylized empirical facts arise in a dynamic capital structure model with investment where firms lack commitment to an ex ante debt policy. Unsecured debt intervention accelerates debt issuance to such an extent that higher potential debt prices are completely offset by increased leverage. Moreover, rather than being used for investment, the proceeds are distributed to shareholders. In contrast, secured debt intervention results in more favorable credit and investment dynamics, even among financially unconstrained firms. Secured debt issuance is disciplined by the collateral constraint, which induces commitment, thus allowing firms to benefit from intervention.

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I thank my committee, Zhiguo He (Co-Chair), Raghuram Rajan (Co-Chair), Stefan Nagel, Fabrice Tourre, and Quentin Vandeweyer, for helpful comments and suggestions.
I also thank Falk Bräuning, José L. Fillat, and Gustavo Joaquim for early discussions on this project while I was a Dissertation Fellow at the Federal Reserve Bank of Boston.

1. Introduction

Over the past decade, central banks have expanded their use of unconventional monetary policy. Notably, they have shown a willingness to directly provide monetary stimulus to risky non-financial corporations through the purchases of corporate bonds. The European Central Bank (ECB) launched the Corporate Sector Purchase Programme (CSPP) in 2016, while the Federal Reserve introduced the Corporate Credit Facilities (CCFs) in 2020. The programs have been immensely successful in reducing the financing costs of targeted firms.[1]

However, the substantial reduction in borrowing costs did not translate into relatively greater investment, a key proxy of real activity, for firms directly targeted by these programs. In both Europe and the United States, firms directly benefiting from corporate bond market stimulus increased leverage and increased payouts to shareholders relative to other firms but did not relatively increase investment.[2] Both the CSPP and CCF directed stimulus to largely financially unconstrained firms rated investment-grade (IG).[3] Both programs were also de facto unsecured corporate bond interventions.[4] Indeed, firms in Europe titled their financing mix toward securities eligible for the CSPP and increased the issuance of unsecured debt (De Santis and Zaghini 2021; Grosse-Rueschkamp, Steffen, and Streitz 2019; Pegoraro and Montagna 2025; Todorov 2020).

[1]Papers documenting the financial effects of the CSPP include Abidi and Miquel-Flores (2018), Pegoraro and Montagna (2025), Todorov (2020), and Zaghini (2019). For the CCFs, papers include Boyarchenko, Kovner, and Shachar (2022); D'Amico, Kurakula, and Lee (2020); Flanagan and Purnanandam (2020); Gilchrist et al. (2021); Haddad, Moreira, and Muir (2021); Kargar et al. (2021); Momin and Li (2022); O'Hara and Zhou (2021)

[2]Papers documenting these dynamics in Europe include De Santis and Zaghini (2021), Grosse-Rueschkamp, Steffen, and Streitz (2019), and Todorov (2020). For the U.S., papers include Darmouni and Siani (2024) and Momin (2025).

[3]Given that eligibility for either program requires the existence of a credit rating, and that the (lack of) availability of credit ratings is a common proxy for financial constraints (e.g. Whited (1992); Almeida, Campello, and Weisbach (2004); Faulkender and Petersen (2006); Denis and Sibilkov (2010); Harford and Uysal (2014)), an extreme argument would be that any intervention in corporate bonds would necessarily direct stimulus to relatively unconstrained firms. Indeed, Greenwald, Krainer, and Paul (2023) make precisely this modeling assumption.

[4]Nearly all of the corporate bonds purchased by both the CSPP and CCF were senior unsecured debt. Detail on security-level CSPP purchases are available here: https://www.ecb.europa.eu/mopo/implement/app/html/index.en.html#cspp. Equivalent data for the CCFs are posted here: https://www.federalreserve.gov/monetarypolicy/smccf.htm.

In this context, this paper is the first to explicitly connect the documented stylized facts of firm dynamics following corporate bond intervention (increased leverage, relative increase in payouts, relative lack of investment response) to the facility design itself (unsecured debt intervention in financially unconstrained firms). This paper rationalizes the empirical patterns observed in the data in a dynamic capital structure model with investment, where firms have access to both equity and (unsecured and secured) debt financing but cannot commit to a leverage policy ex ante, in the vein of Demarzo and He (2021). The model is numerically estimated with parameters taken from the literature and is shown to fit key empirical moments. It is further used to show how (counterfactual) secured debt intervention, rather than unsecured debt intervention, can induce a stronger investment response among financially unconstrained firms. To the extent that central banks intervene in corporate debt markets to stimulate real activity, this is an important consideration for policy design.[5]

While the arguments presented in this paper are novel, the dynamics involving unsecured debt intervention are also present in the model of Crouzet and Tourre (2021), which this paper builds on. Crouzet and Tourre (2021) extend the model of Demarzo and He (2021) to feature investment, subject to convex adjustment costs, that follow 'q-theory' dynamics (Hayashi 1982). Hence, the investment rate is proportional to the marginal value of equity. Without commitment, the firm cannot realize the tax shield benefits of debt issuance. Bond investors anticipate the equilibrium leverage policy of equity shareholders and price in higher default costs, exactly offsetting any gains to firm value from the prescence of a debt tax shield.

Likewise, unsecured debt intervention results in accelerated debt issuance which is paid out by the firm to shareholders, leaving the firm with higher leverage. Higher leverage, and hence, bankruptcy costs, imply a lower continuation value. These two forces (higher payouts and higher bankruptcy costs) cancel out to leave firm equity value, as well as investment, unchanged.[6] However, longer-run investment dynamics suffer due to higher firm leverage (as implied by the lower continuation value).[7]

[5]Both the ECB and Fed emphasize how loosening financial conditions are expected to support real activity in their announcements of corporate bond purchase programs. For the ECB's announcement, see https://eur-lex.europa.eu/legal-content/EN/TXT/PDF/?uri=CELEX:32016D0016. For the Fed's, see https://www.federalreserve.gov/newsevents/pressreleases/monetary20200323b.htm.

[6]DeMarzo, He, and Tourre (2023) document similar dynamics in the context of a risk-neutral sovereign borrower and more patient international creditors.

[7]This prediction also has some empirical support. Momin (2025) finds that while firm leverage

While stark, I emphasize this mechanism as an explanation for the stylized firm dynamics seen in both Europe and the United States following the introduction of corporate bond purchase programs.

I extend the model of Crouzet and Tourre (2021) to feature secured debt that is issued subject to a non-state contingent collateral constraint, similar to Kiyotaki and Moore (1997). Secured debt, unlike unsecured debt, induces commitment via this collateral constraint, echoing the finding in Demarzo (2019). I show that the value of the Lagrange multiplier on the collateral constraint is precisely equal to the marginal value of the debt tax shield. Intuitively, the firm can enjoy the benefits of the debt tax shield because the issuance of fully collateralized secured debt is limited by the collateral constraint, which prevents the firm from diluting creditors through excessive debt issuance, as was the case with unsecured debt issuance, given a lack of firm commitment to an ex ante debt policy.

Given that fully collateralized secured debt is risk-free, while firms benefit from the debt tax shield, firms issue up to the collateral constraint, which thus binds. The result that firms exhaust debt capacity is seemingly at odds with Rampini and Viswanathan (2010), who model state-contingent collateral constraints and show that firms engage in risk management by maintaining financial slack for future states. Secured debt is also risk-free in Rampini and Viswanathan (2010), but the key difference is that firms are subject to additional financial constraints in the form of restrictions on equity issuance. In contrast, I maintain the standard Leland (1994) assumptions in my baseline model that firms can access both debt and equity markets, so long as its continuation value is non-negative. When equity issuance constraints are removed from Rampini and Viswanathan (2010), I recover the result that the collateral constraint binds for all states.[8]

Empirically, firms tapped credit lines, issued corporate bonds, and issued equity through the COVID-19 pandemic.[9] This supports the modeling choice to maintain

rose following the introduction of the CCFs, investment rates have not exceeded their pre-pandemic levels through 2023, in the case when investment is proxied as the change in gross property, plant, and equipment (PP&E).

[8]See Section 2.3 for the discussion and Section 6.5 in the Appendix for the proof.

[9]Papers documenting credit line drawdowns include: Acharya and Steffen (2020); Darmouni and Siani (2024); Greenwald, Krainer, and Paul (2023). Similarly, for bond issuance: Becker and Benmelech (2021); Boyarchenko, Kovner, and Shachar (2022); Darmouni and Siani (2024); Dutordoir et al. (2024); Halling, Yu, and Zechner (2020); Hotchkiss, Nini, and Smith (2022). And for equity issuance: Dutordoir et al. (2024); Halling, Yu, and Zechner (2020); Hotchkiss, Nini, and Smith (2022)

the standard Leland (1994) assumptions. However, this results in counterfactual dynamics for the collateral constraint and secured debt issuance, where the collateral constraint always binds and secured debt issuance is procyclical, absent intervention. Empirically, secured debt issuance is countercyclical, with firms maintaining slack in the collateral constraint and maintaining financial flexibility for 'bad' states as a form of insurance (Benmelech, Kumar, and Rajan 2022, 2024). Ultimately, this translates into a more conservative modeling choice that underestimates the efficacy of potential secured debt intervention, which otherwise has more impact in economies with more freely available collateral.

Despite this, I find that secured debt intervention, which entails public lending against collateral valued above market prices,[10] boosts firm investment through direct and indirect channels. First, secured debt intervention makes secured debt issuance more valuable, directly incentivizing the firm to invest and raise collateral to relax its collateral constraint. Second, because secured debt issuance has implicit commitment, the firm is able to benefit from greater proceeds from fully collateralized secured debt issuance, without increasing default risk. This results in a higher equity value, as well as a higher value of Tobin's q. Since capital is perceived as more productive, investment indirectly increases, as well.

In terms of longer-term dynamics, secured debt intervention leads to higher expected average equity prices, debt prices, investment rates, and lower default rates, relative to the case of no intervention and especially, compared to the case of unsecured debt intervention. Since unsecured debt intervention accelerates debt issuance and leads firm to accumulate leverage, the longer-term dynamics are actually more unfavorable compared to the benchmark of no intervention.

This motivates studying unsecured debt intervention with payouts limited by dividend restrictions, which generates higher investment, higher unsecured debt prices, and lower default rates. However, it induces firms to repurchase debt and also leads to lower equity valuations. Restricting debt repurchases further improves investment dynamics (though, it does not improve equity valuations) and firms choose not to issue unsecured debt when they would have otherwise repurchased debt. All together, the numerical solutions suggest firms would not voluntarily participate in an unse-

[10] The nature of the secured debt intervention described here is analogous to the Bank Term Funding Program (BTFP) administered by the Fed to provide loans to financial institutions against collateral valued at par (hence, above market prices): https://www.federalreserve.gov/newsevents/pressreleases/files/monetary20240124a1.pdf.

cured debt intervention program with dividend restrictions.

This paper contributes to the extensive literature on the financial and real effects of central bank corporate bond purchase programs referenced earlier in this section. To the best of my knowledge, this is the first paper to connect the stylized empirical facts of the real effects of these programs to the nature of the intervention itself (unsecured debt intervention in financially unconstrained firms). It is also the first to suggest that secured debt intervention, rather than unsecured debt intervention, would have improved investment outcomes. Crouzet and Tourre (2021) take a broader view on corporate credit interventions to include the Main Street Lending Program (MSLP) and Paycheck Protection Program (PPP), which featured subsidized bank lending to smaller, generally non-rated firms. They find that credit interventions can prevent inefficient firm restructurings during a credit shock, quantitatively dominating longer-run drags on investment due to debt overhang. Li and Li (2024) also broadly analyze corporate credit programs and highlight the potential negative long-run implications of rescuing low-quality firms that exacerbates future intervention costs. Greenwald, Krainer, and Paul (2023) develop a structural corporate finance model with bank term loans, credit lines, and corporate bonds. Interestingly, they also find that corporate bond intervention generates higher corporate bond issuance by the financially unconstrained firms that issue them, largely without generating additional investment. However, they show that such an intervention can still indirectly stimulate investment by freeing up bank credit lines for more constrained firms.

This paper also contributes to the literature on dynamic capital structure models where firms lack commitment to an ex ante policy. Demarzo and He (2021) build on the seminal work of Leland (1994) to show that firms cannot benefit from the debt tax shield when they lack commitment, although firms still issue debt in equilibrium. DeMarzo, He, and Tourre (2023) explore the ramifications of Demarzo and He (2021) in the context of sovereign debt, while Crouzet and Tourre (2021) extends the model to include continuous investment policies subject to convex adjustment costs. They structurally estimate their model and show that the model-implied moments align well with key empirical moments. I further extend Crouzet and Tourre (2021) to include both continuous investment subject to convex adjust costs as well as secured debt[11] and show that the numerically estimated model delivers similar quantitative

[11] Demarzo (2019) first models secured debt in the context of Leland-type models without commitment. There, investment opportunities arrive according to a Poisson process and do not follow q-theory dynamics.

performance.

The third strand of literature this paper contributes to is on the effects of accommodative monetary policy on leveraged payouts. Elgouacem and Zago (2023) show empirically that firms finance share buybacks by issuing corporate bonds. They find that accommodative monetary policy increases buybacks. Acharya and Plantin (2025) also note that the increase in firm payouts have occurred a low-yield environment where the corporate bond market has significantly expanded, while investment has remained depressed. They rationalize these dynamics in a parsimonious model featuring agency frictions and moral hazard that arise due to the increasing relationship between investment returns and shareholders' costly private effort. Pazarbasi (2025) shows empirically that cash-rich firms have higher equity payouts and finds that this can be explained in a New Keynesian model where accommodative monetary policy reduces firms' precautionary cash demand, triggering payouts.

The rest of the paper is organized as follows. Section 2 sets up the model with short-term secured debt. Section 3 explores crisis dynamics in the model. Section 4 presents the numerical solution to the model. Section 5 concludes.

2. Model with Short-Term Secured Debt

2.1. Setup

In the baseline model with short-term secured debt, I assume shareholders and creditors are risk-neutral with discount rate r. Shareholders have the option to default on both unsecured and secured creditors at any point. At default, I assume shareholders and unsecured creditors have zero recovery value, while secured creditors receive the collateral backing their debt. Following the standard assumption in Leland models, equity investors are deep-pocketed and can support the firm with liquidity injections. As a result, firms can be viewed as financially unconstrained, given their ability to raise debt and equity to finance operations, so long as its continuation value is positive.

Firms' production technology, in revenue per unit of time, is given by:

$$Y_t = AK_t$$

where the productivity parameter A is deterministic. Capital, K_t, is measured in efficiency units and follows a geometric Brownian motion given by:

$$\frac{dK_t}{K_t} = (g_t - \delta)dt + \sigma dZ_t$$

where g_t is the endogenous investment rate and δ is the capital depreciation rate, where $\delta \in (0, 1)$. dZ_t is the increment of a Brownian motion and is distributed as $dZ_t \sim \mathrm{N}(0, dt)$. The price of capital is fixed at 1, as in Crouzet and Tourre (2021).

Shareholders choose a rate of investment, subject to a convex cost, parameterized as:

$$\Phi(g) = \frac{1}{2}\gamma g^2$$

This parameterization of investment costs ensures investment is nonnegative in equilibrium (and hence, capital is never liquidated). This is in contrast to Crouzet and Tourre (2021), where the firm liquidates its capital at points over the state space (i.e. $g < 0$).[12]

The firm's stock of unsecured debt has an aggregate face value F_t and is an endogenous state variable. Unsecured debt matures at a Poisson rate m^u and has a price p_t. It is issued at a face value equal to 1 with a coupon equal to the risk-free rate, $c^u = r$. Given potential default risk, $p_t \leq 1$. Unsecured debt stock evolves as:

$$dF_t = \underbrace{-m^u F_t dt}_{\text{maturing debt}} + \underbrace{d\Gamma_t^u}_{\text{active debt management}}$$

Following Demarzo and He (2021), I focus on a 'smooth' equilibrium where endogenous unsecured debt is assumed to be continuous. Hence, $d\Gamma_t^u = B_t^u dt$, where B_t^u is the endogenous unsecured debt issuance policy.

Secured debt has an aggregate face value of S_t and is another endogenous state variable. Short-term secured debt is assumed to mature instantaneously with maturity dt. When issued at par with face value equal to 1 and paying a coupon equal to

[12]The chief motivation for this modification is to ensure that long-term secured debt is risk-free in the extension to the model examined in Section 6.3 of the Appendix, keeping the modeling tractable. With instantaneously maturing 'short-term' secured debt, the level of the secured debt adjusts with capital liquidation, so this parameterization is not critical. With long-term secured debt, capital liquidation would cause initially fully collateralized debt to become under-collateralized, absent restrictions on the firm from doing so.

the risk-free rate, $c^s = r$, short-term secured debt is risk-free with price equal to 1.[13] Let $S_{t^-} \equiv \lim_{dt \searrow 0} S_{t-dt}$ be the value of secured debt issued at the instant before time t, then secured debt evolves as $dS_t = S_t - S_{t^-} \equiv B^s_t dt$.

Additionally, I assume that the firm faces a non-state contingent collateral constraint when issuing secured debt that is proportional to its capital stock:

$$S_t \leq \alpha K_t$$

where $\alpha \in (0, 1)$ is the proportion of the capital stock pledgeable as collateral to secured creditors.

2.2. Equity's Problem

Let θ equal the corporate tax rate. Then, equity's flow payoffs are:

$$\Big[\underbrace{AK_t}_{\text{revenue}} - \underbrace{\theta(AK_t - c^u F_t - c^s S_{t^-})}_{\text{corporate taxes}} - \underbrace{\Phi(g_t)K_t}_{\text{investment cost}}$$
$$- \underbrace{(c^u + m^u)F_t}_{\text{unsecured debt interest \& principal}} + \underbrace{p_t B^u_t}_{\text{unsecured debt net issuance}}$$
$$- \underbrace{c^u S_t}_{\text{secured debt interest}} + \underbrace{B^s_t}_{\text{secured debt net issuance}} \Big] dt$$

Shareholders maximize the present discounted cash flows, taking unsecured debt price, p_t, as given, and choose policies for investment, unsecured debt issuance, secured debt issuance (subject to collateral constraint), and default time, τ. The sequence formulation of the stochastic control and optimal stopping problem is:

$$J(K, F, S) = \max_{\tau, g, B^u, B^s} \mathbb{E}_0 \Big[\int_0^\tau \exp(-rt)[AK_t - \theta(AK_t - c^u F_t - c^s S_t) - \Phi(g_t)K_t$$
$$-(c^u + m^u)F_t + p_t B^u_t - c^s S_t + B^s_t]dt \Big| K_0 = K, F_0 = F, S_0 = S \Big] \tag{1}$$

[13] In this setup, instantaneously maturing debt is risk-free so long as priced shocks to the income/production process are continuous, which is the case when shocks are determined by increments of a Brownian motion (DeMarzo, He, and Tourre 2023; Hu, Varas, and Ying 2024). However, this is not necessarily the case if there is priced jump risk (Abel 2016, 2018; Hu, Varas, and Ying 2024). In the presence of jump risks, instantaneously maturity debt becomes risk-free if fully collateralized Abel (2018). See Section 2.3 for further discussion.

$$s.t.$$

$$\frac{dK_t}{K_t} = (g_t - \delta)dt + \sigma dZ_t$$

$$dF_t = -m^u F_t dt + B_t^u dt$$

$$dS_t = B_t^s dt$$

$$S_t \leq \alpha K_t$$

As noted by Abel (2018), the value of shareholders' equity is given by $J - S$, where S is the value of short-term debt. However, in solving for optimal policies, shareholders jointly maximize the value of equity and short-term creditors because they immediately receive the proceeds from the issuance of short-term debt.[14]

To show that the value function is homogeneous of degree 1 in K, note that capital is given by:

$$K_t = K_0 \exp\left(\int_0^t \left(g_t - \delta - \tfrac{1}{2}\sigma^2\right) dt + \int_0^t \sigma \, dZ_t \right).$$

Then, substitute this expression into the firm's objective given by Equation (1) and factor out $K_0 = K$. Rescale the state variables and controls by K_t:

$$f_t \equiv \frac{F_t}{K_t}, \quad s_t \equiv \frac{S_t}{K_t}, \quad b_t^u \equiv \frac{B_t^u}{K_t}, \quad b_t^s \equiv \frac{B_t^s}{K_t},$$

Under the change of measure $dZ_t \equiv d\tilde{Z}_t + \sigma dt$, the evolution of the state variables are given by:

$$df_t = [b_t^u - (g_t - \delta + m^u) f_t dt] - \sigma f_t d\tilde{Z}_t$$

$$ds_t = [b_t^s - (g_t - \delta)] s_t dt - s_t \sigma d\tilde{Z}_t$$

Then, the rescaled value function is given by:

(2)

$$j(f, s) = \max_{\tau, g, b^u, b^s} \tilde{\mathbb{E}}_0 \left[\int_0^\tau \exp\left(- \left(r - \int_0^t g_s ds + \delta \right) t \right) [A - \theta(A - c^u f_t - c^s s_t) - \Phi(g_t) - (c^u + m^u) f_t + p_t b_t^u - c^s s_t + b_t^s] dt \middle| f_0 = f, s_0 = s \right]$$

[14]See also Hu, Varas, and Ying (2024) for similar arguments.

$$s.t.$$
$$df_t = [b_t^u - (g_t - \delta + m^u)f_t]dt - f_t \sigma d\tilde{Z}_t$$
$$ds_t = [b_t^s - (g_t - \delta)s_t]dt - s_t \sigma d\tilde{Z}_t$$
$$s_t \leq \alpha$$

Consequently, the Hamilton-Jacobi-Bellman (HJB) equation characterizing equity's problem in the continuation region is given by:

(3)
$$0 = \max_{g, b^u, b^s} \Bigg\{ -(r - g + \delta) j - (s - \alpha) l^s \underbrace{+A - \theta(A - c^u f - c^s s) - \Phi(g) - (c^u + m^u) f + p b^u - c^s s + b^s}_{\text{cash flows per unit of capital}} \underbrace{+[b^u - (g - \delta + m^u) f] j_f + \frac{1}{2}\sigma^2 f^2 j_{ff}}_{\text{evolution of unsecured debt per unit of capital}} \underbrace{+[b^s - (g - \delta) s] j_s + \frac{1}{2}\sigma^2 s^2 j_{ss}}_{\text{evolution of secured debt per unit of capital}} \Bigg\}$$

where l^s is the Lagrange multiplier on the collateral constraint.

PROPOSITION 1 (Collateral Constraint Binds). *The collateral constraint binds, and the HJB equation becomes:*

(4)
$$0 = \max_{g, b^u} \Bigg\{ -(r - g + \delta) j + A - \theta(A - c^u f - c^s \alpha) - \Phi(g) - (c^u + m^u) f + p b^u - c^s \alpha + \alpha(g - \delta) +[b^u - (g - \delta + m^u) f] j_f + \frac{1}{2}\sigma^2 f^2 j_{ff} \Bigg\}$$

Proof. See Section 6.2 in the Appendix for the full derivation. The solution method can be summarized as:

(a) Take derivative of equity's HJB in continuation region with respect to secured debt issuance policy $\rightarrow$ obtain equilibrium derivative conditions for equity value function.

(b) Take derivative of HJB with respect to s and use equilibrium derivative conditions. Obtain Lagrange multiplier on constraint.

(c) Assuming Lagrange multiplier binds, substitute in value for s and use equilibrium derivative conditions to obtain equity HJB reduced by 1 state variable and 1 control variable.

□

COROLLARY 1 (Value of Commitment). *The value of commitment, as suggested by the Lagrange multiplier, is equal to the marginal value of the debt tax shield ($l^S = \theta c^S$). That is, relaxing the collateral constraint and allowing the firm to issue additional secured debt generates additional tax shield benefits.*

Proof. Follows from the proof of Proposition 1. □

Equivalent results to Proposition 1 and Corollary 1 hold in the case when the firm can issue long-term riskless secured debt, as shown in Appendix 6.3. That is, the collateral constraint binds, and the value of the Lagrange multiplier equals the debt tax shield. However, modeling secured debt as instantaneously maturing simplifies the numerical solution and discussion of the results.[15]

The ability of collateral to induce commitment, allowing the firm to enjoy the tax shield benefits from issuing secured debt was first made by Demarzo (2019), in the context of Leland-type models. This contrasts with the main finding in Demarzo and He (2021), that absent commitment to an ex ante debt policy, the firm cannot monetize the tax shield benefits from issuing unsecured debt, as shown in Section 2.5. This is because unsecured creditors immediately discount the price of unsecured debt by the same value as the debt tax shield, owing to the larger bankruptcy costs engendered by the additional issuance of risky, unsecured debt.

2.3. Discussion on Collateral Constraint and Financial Slack

I model collateral constraints in a non-state contingent fashion, as in Kiyotaki and Moore (1997), and obtain a similar result that the collateral constraint binds when there is a motivation to trade, either due to a difference in discount rates or the presence of a debt tax shield. In the models considered here, both short-term and long-term secured debt are risk-free. As such, the marginal cost of issuing secured debt is zero, since there is no impact from increased exposure to bankruptcy. At the same time, the marginal benefit is the interest tax shield. As a result, firms issue up to their collateral constraint, completely exhausting debt capacity.

[15] Future extensions of this model can model risky, long-term secured debt where risk is driven by fluctuating capital quality or capital prices.

Rampini and Viswanathan (2010) feature state-contingent debt and collateral constraints, where firms also face non-negativity constraints on dividends. All agents in the model have the same discount rate and there are no taxes. Firms can borrow in the current state by issuing promises to pay in future states, subject to the collateral constraint, which ensures that debt is risk-free. The benefit to issuing debt is the expected returns to investment in the current and future period, while the cost is the expected return from conserving net worth and increasing investment in certain future states. Consequently, the authors show that latter may dominate, leading firms to maintain slack in the collateral constraint in some states.

However, as shown in Appendix 6.5, when the Rampini and Viswanathan (2010) environment is modified to allow dividends to be unconstrained, so that the firm can receive cash infusions from equity investors, as is the case in the model presented in this section, and a debt tax shield is introduced, one recovers the result that collateral constraints bind, even with state-contingency. Removing the nonnegativity constraints on dividends eases the firm's financial constraints. Since debt is risk-free, and the firm receives benefit from the debt tax shield, the firm issues up to the collateral constraint.

Empirically, firms issued both corporate bonds, as well as, equity during the COVID-19 pandemic.[16] Moreover, as argued in Section 1, corporate bond purchase programs generally directed monetary stimulus to relatively unconstrained firms (IG-rated firms). All together, the standard assumptions of Leland-type models seem more appropriate.

As noted by Hu, Varas, and Ying (2024), unsecured short-term debt is risk-free whenever shocks to the firm's earnings process is governed by an Itô process, as is the case in the model presented in this section. Consequently, if firms have a motive to borrow due to a difference in discount rates or a debt tax shield, they will exhaust the borrowing capacity imposed by limited liability. To induce risky unsecured short-term debt, Abel (2018) and Hu, Varas, and Ying (2024) introduce possible downward jumps in firm earnings. With this modification, issuing unsecured short-term debt can expose the firm to potential bankruptcy costs and so, the firm may choose to not exhaust its borrowing capacity. However, if short-term debt is fully collateral-

[16] Papers documenting record bond issuance include: Becker and Benmelech (2021); Boyarchenko, Kovner, and Shachar (2022); Darmouni and Siani (2024); Dutordoir et al. (2024); Halling, Yu, and Zechner (2020); Hotchkiss, Nini, and Smith (2022). Papers documenting equity issuance, particularly for more financially constrained firms, include: Dutordoir et al. (2024); Halling, Yu, and Zechner (2020); Hotchkiss, Nini, and Smith (2022)

ized, then it is risk-free even in the presence of jumps, and the firm will exhaust its borrowing capacity.[17]

Empirically, secured debt issuance is countercyclical, with firms maintaining slack in the collateral constraint, maintaining financial flexibility for 'bad' states as a form of insurance (Benmelech, Kumar, and Rajan 2022, 2024). More realistic model dynamics would necessitate generating a slack in the collateral constraint, which could be done by introducing frictions to collateralizing capital, issuing secured debt, etc. However, this would imply a stronger response from secured debt intervention in the crisis period (characterized in Section 3) as firms would enter this state with greater capacity to issue secured debt. Indeed, a firm's limited capacity to issue secured debt is precisely what disciplines and enables it to benefit from secured debt intervention; in contrast, unsecured debt intervention debt accelerates issuance to the point where any potential benefits are exactly offset by higher bankruptcy costs incurred from greater indebtedness. Consequently, a more conservative and parsimonious modeling approach is followed where the collateral constraints bind.

2.4. Unsecured Creditors' Problem

Unsecured creditors take equity's optimal policies as given and price unsecured debt rationally (i.e. anticipating future default). The price of 1 unit of unsecured debt with face value 1 is given by:

$$p(K,F) \equiv \mathbb{E}_0\left[\int_0^{\tau} \exp(-(r+m^u)t)(c^u+m^u)dt \middle| K_0 = K, F_0 = F\right]$$

$$s.t.$$

$$\frac{dK_t}{K_t} = (g_t - \delta)dt + \sigma dZ_t$$

$$dF_t = -m^u F_t dt + B_t^u dt$$

The unsecured debt price is homogeneous of degree zero in capital: $p(1, F/K) = p(f)$. Given the drift for df_t, the HJB for the debt value function in the continuation region

[17]This can be seen in (Abel 2018, p. 104) Equation (11) by setting the fraction of deadweight losses from default, α, equal to zero. Then, the derivative of the trade-off function with respect to debt is equal to the debt tax shield and strictly positive.

is given by:

$$(5) \qquad (r+m^u)\,p = c^u + m^u + [b^u - (g + m^u - \delta - \sigma^2) f]\, p_f + \frac{1}{2}\sigma^2 f^2 p_{ff}$$

2.5. Optimal Policies

The first order condition for the HJB equation shown in Equation (4) with respect to g yields:

$$\Phi'(g) = j - f j_f + \alpha$$
$$g = \frac{1}{\gamma}(j - f j_f + \alpha)$$

Tobin's q is given by the marginal derivative of J with respect to K (with the price of capital fixed at one):

$$q \equiv \partial_K J = \frac{\partial J(K,F)}{\partial K} = \frac{\partial (K j(f = F/K))}{\partial K}$$
$$= j(f) - f j_f(f)$$

With a binding collateral constraint, collateralized borrowing provides a direct motivation to invest in order to relax the collateral constraint. Hence, the ability to monetize a portion of the capital stock through secured debt issuance increases optimal investment by a factor proportional to α, relative to Tobin's q.

Similarly, Abel (2016) finds that the availability of instantaneously maturity, short-term bond financing boosts firm investment indirectly via increasing the joint value of equity and short-term debt. In contrast, this channel is absent in Crouzet and Tourre (2021) who only allow firms to issue unsecured debt. As shown below, absent commitment, firms do not benefit from unsecured debt issuance.

The FOC of equity's problem with respect to b^u:

$$\underbrace{p}_{\text{MB of issuance}} + \underbrace{j_f}_{\text{MC on future value}} = 0$$

This holds for all equilibria; in particular, it holds for the no-trade equilibrium where equity does not issue unsecured debt (i.e. $b^u = 0$). The economic content of this result is that the marginal benefit equity gains from issuing debt (the amount raised) is completely offset by the marginal impact on equity due to higher unsecured debt levels. Stated otherwise, lenders anticipate the firms issuance policy and price in higher default risk at issuance. Consequently, this allows one to solve for the equity value assuming no-trade.

Nonetheless, while equity value is unaffected by unsecured debt issuance, in equilibrium, equity does issue unsecured debt, given the presence of the debt tax shield. Likewise, unsecured creditors require knowledge of this policy to accurately price unsecured debt; otherwise, the equilibrium condition $p = -j_f$ does not hold.

To solve for the optimal issuance policy, first take the derivative of the HJB equation characterizing equity's problem without trade (setting $b^u = 0$ in Equation (4) and using the envelope theorem with respect to optimal investment):

$$\begin{aligned}(r - g + \delta) j_f &= \theta c^u - (c^u + m^u) - (g - \delta + m^u) j_f - (g - \delta + m^u) f j_{ff} + \sigma^2 f j_{ff} + \frac{1}{2}\sigma^2 f^2 j_{fff} \\ \Rightarrow (r + m^u) j_f &= \theta c^u - (c^u + m^u) - (g - \delta + m^u - \sigma^2) f j_{ff} + \frac{1}{2}\sigma^2 f^2 j_{fff}\end{aligned}$$

Then, substitute in the equilibrium condition $p = -j_f$ into Equation (5):

$$-(r + m^u) j_f = c^u + m^u - [b^u - (g + m^u - \delta - \sigma^2) f] j_{ff} - \frac{1}{2}\sigma^2 f^2 j_{fff}$$

Combine these two results and obtain the optimal unsecured debt issuance rate:

$$\begin{aligned}0 &= \theta c^u - b^u j_{ff} \\ \Rightarrow b^u &= \frac{\theta c^u}{j_{ff}} = \frac{\theta c^u}{-p_f} > 0\end{aligned}$$

Given a strictly convex value function for equity (see Proposition A1 in the Appendix) and short-term debt ($j_{ff} > 0$), unsecured debt issuance is strictly positive in the continuation region (outside of default).

Taking the derivative of unsecured debt issuance policy with respect to f yields:

$$b^u_f = \frac{\theta c^u}{p_f^2} p_{ff}$$

Thus, the monotonicity of unsecured debt issuance depends on the convexity or concavity of debt prices. If debt prices are convex, then unsecured debt issuance increases with leverage; if they are concave, it decreases. The convexity or concavity of debt prices depends on the parameter values used in estimating the model.

3. Crisis Dynamics with Short-Term Debt

A crisis is modelled as an unforeseen shock which causes productivity, A, to drop to ηA, where $\eta < 1$. The economy jumps back to its pre-shock equilibrium at the Poisson rate λ so that the expected length of the crisis is $1/\lambda$.

Additionally, let p_s^* equal the exogenous price of 1 unit of short-term secured debt during a crisis. If $p_s^* < 1$, then this is equivalent to investors demanding a haircut when lending to firms against their collateral, similar to repo haircuts. Short-term secured debt is issued at a premium if $p_s^* > 1$, which may be the case in the event of intervention. Note that absent an exogenous change in price, the endogenous price of short-term secured debt would still be risk-free with price equal to one.

PROPOSITION 2 (Crisis HJB with Binding Constraint). *The collateral constraint binds in a crisis regime, if p_s^* satisfies:*

$$\frac{\theta r}{1+r+\lambda} > 1 - p_s^* \tag{6}$$

That is, if the discounted value of the debt tax shield is greater than the haircut, the collateral constraint binds.

Given p_s^* so that the collateral constraint binds, the crisis joint equity and short-term debt HJB in the continuation region is:

$$\begin{aligned} 0 = \max_{b^u, g} \Big\{ & -(r - g + \delta + \lambda) j + \eta A - \theta(\eta A - c^u f - c^s \alpha) - \Phi(g) \\ & - (c^u + m^u) f + p b^u - c^s \alpha + \alpha(p_s^* - 1) + p_s^* \alpha(g - \delta) + \lambda \bar{j} \\ & + [b^u - (g + m^u - \delta) f] j_f + \frac{1}{2}\sigma^2 f^2 j_{ff} \Big\} \end{aligned} \tag{7}$$

where $\bar{j}$ is the pre-shock value of equity and short-term debt. Note that $\bar{j}$ can be solved independently of j.

Proof. See Section 6.6 in the Appendix. □

The first order condition with respect to g is:

$$\begin{aligned} 0 &= j - f j_f - \Phi'(g) + p_s^* \alpha \\ \Rightarrow \Phi'(g) &= j - f j_f + p_s^* \alpha \end{aligned}$$

Thus, the exogenous price of secured debt directly impacts investment policy, while also indirectly affecting it through changes in the value function and hence, q.

Additionally, note that the crisis HJB for unsecured debt price is given by:

$$(r + m^u)\, p = c^u + m^u + [b^u - (g + m^u - \delta - \sigma^2) f]\, p_f + \frac{1}{2}\sigma^2 f^2 p_{ff} + \lambda(\bar{p} - p) \tag{8}$$

where $\bar{p}$ is the pre-shock unsecured debt price consistent with $\bar{j}$.

3.1. Secured Debt Intervention

PROPOSITION 3 (Secured Debt Intervention Strictly Increases Equity Value). *Consider an intervention in secured debt that results in an increase in the price of secured debt to a level higher than before intervention, such that Equation 6 is satisfied. As a result, the collateral constraint binds and there is a strict increase in both the joint value of equity and short-term debt, and the value of equity.*

Proof. See Section 6.7 in the Appendix. □

Proposition 3 states that if secured debt intervention, which is modeled as the central bank purchasing secured debt at a premium relative to the prevailing market price,[18] is sufficiently high such that firms have an incentive to issue secured debt (i.e. Proposition 2 holds), then both the joint value of equity and short-term debt as well as just the value of equity are strictly increased. The result is not surprising, since the firm receives more proceeds from secured debt issuance when prices are higher, on the intensive margin, and on the extensive margin, sufficiently high secured debt prices make issuance worthwhile for the firm. Given these results, secured debt intervention is modeled as a premium offered above par in the numerical estimation, for simplicity.

PROPOSITION 4 (Secured Debt Intervention Strictly Increases Investment). *Provided that Proposition 3 holds, investment is strictly increasing in the amount of secured debt intervention. Furthermore, the increase can be decomposed into direct and indirect channels.*

- *Direct Channel: Higher proceeds from secured debt issuance directly boost investment by strengthening the motivation for the firm to investment and relax the collateral constraint.*

[18] The nature of the secured debt intervention described here is analogous to the Bank Term Funding Program (BTFP) administered by the Fed to provide loans to financial institutions against collateral valued at par (hence, above market prices): https://www.federalreserve.gov/newsevents/pressreleases/files/monetary20240124a1.pdf.

- *Indirect Channel: Higher value of Tobin's q, i.e. higher marginal value of capital, indirectly boosts investment.*

Proof. See Section 6.8 in the Appendix. □

Proposition 4 states that secured debt intervention strictly increases investment. Moreover, the higher the premium offered in the intervention, the higher the increase in investment. Secured debt intervention increases investment directly and indirectly. First, by making secured debt issuance more lucrative, intervention directly increases the motivation of the firm to invest and relax its collateral constraint. Second, secured debt intervention increases the value of Tobin's q, implying a higher marginal value of capital, thus indirectly stimulating investment.

3.2. Unsecured Debt Intervention

As in Crouzet and Tourre (2021), I model an unsecured debt intervention as the government becoming the marginal buyer in the unsecured debt market during a crisis period. This results in segmented equity and credit markets. Hence, we would have different discount rates corresponding to equity investors, $r^{(e)}$, and unsecured debt investors, $r^{(d)}$.

PROPOSITION 5 (Unsecured Debt Intervention Accelerates Issuance). *In the case where unsecured debt intervention segments equity and credit markets with the government becoming the marginal buyer of debt, the endogenous unsecured debt issuance policy becomes:*

$$b^u = \frac{\theta c^u}{-p_f} + \frac{(r^{(e)} - r^{(d)})\, p}{-p_f}$$

and is increasing in the wedge between the discount rates of equity investors, $r^{(e)}$, and the discount rate implied by the government's subsidy, $r^{(d)}$. Unsecured debt intervention implies $r^{(d)} < r^{(e)}$ and hence, higher issuance in the continuation region since p is decreasing in leverage and $p > 0$ outside of default.

Proof. See Section 6.9 in the Appendix. □

The nature of the unsecured debt intervention considered in Proposition 5 leads to an equilibrium increase in firms' unsecured debt issuance, but does not imply a higher price for unsecured debt. Indeed, the potential improvement in unsecured debt price implied by government intervention at lower discount rates is exactly offset by accelerated issuance which leaves unsecured debt price unchanged. This is a stark consequence of a lack of commitment to ex ante debt policy. Empirically, debt issuance does not completely offset the price impact of unsecured debt intervention, although it does accelerate significantly.

PROPOSITION 6 (Unsecured Debt Intervention Accelerates Firm Payouts to Shareholders). *Proposition 5 shows that unsecured debt intervention accelerates the issuance of unsecured debt, while the price of unsecured debt is unaffected due to the intervention. This results in higher proceeds from unsecured debt issuance. Firms do not use these proceeds for investment but rather pay these out to shareholders. Higher payouts are exactly offset by a lower joint continuation value for equity and short-term debt due to higher default costs induced by higher leverage.*

Proof. See Section 6.10 in the Appendix. □

Proposition 6 implies that unsecured debt intervention increases firm leverage and defaults, relative to non-intervention, a prediction which is verified in the numerical estimation. While unsecured debt intervention does not affect the discounted equity price, since higher payouts and higher default costs due to higher leverage are exactly offset, it does imply negative dynamics for future investment.

3.3. Dividend Restriction

An alternative method to induce commitment is the use of dividend restrictions which prevents unsecured debt issuance from immediately being paid out as dividends. While a permanent dividend restriction will cause equity price to fall to 0, a temporary dividend restriction can be implemented during the crisis period while still maintaining positive equity prices. Given that unsecured debt intervention distorts firm incentives to accelerate issuance to increase payouts to shareholders, a natural question is whether this force can be curtailed through the use of dividend restrictions, causing firms to increase investment instead.

PROPOSITION 7 (Optimal Policies with Dividend Restrictions). *With dividend restrictions, investment (g) and unsecured debt issuance policies (b^u) satisfy:*

$$\begin{aligned}\pi(b^u, g) \equiv & \eta A - \theta(\eta A - c^u f - c^s\alpha) - \Phi(g) \\ & - (c^u + m^u) f + p b^u - c^s\alpha + \alpha(p_s^* - 1) + p_s^*\alpha(g - \delta) \leq 0\end{aligned} \tag{9}$$

Let l be the Lagrange multiplier on this constraint, where $l \geq 0$, with equality when the constraint is slack. Taking l as given, the crisis joint equity and short-term debt HJB with a temporary dividend restriction is:

$$(10)\quad 0 = \max_{b^u, g} \left\{ -(r - g + \delta + \lambda) j + (1 - l)\pi(b^u, g) + \lambda \bar{j} + [b^u - (g - \delta + m^u) f] j_f + \frac{1}{2}\sigma^2 f^2 j_{ff} \right\}$$

The expressions for unsecured debt price, unsecured debt issuance, and investment are given by:

$$p = -\frac{j_f}{1 - l}$$

$$g = \frac{1}{\gamma}\left(\frac{j - f j_f}{1 - l} + p_s^* \alpha\right)$$

$$b^u = (1 - l)\frac{\theta c^u}{j_{ff}} + l \frac{\lambda \bar{j}_f}{j_{ff}}$$

Proof. See Section 6.11 in the Appendix. □

As a result of the dividend constraint, it is no longer the case that equity prices are invariant to unsecured debt policy. Additionally, given $l \in [0, 1)$, relative to the unconstrained case $l = 0$, unsecured debt price is higher, investment is higher, and unsecured debt issuance is lower. In particular, Section 6.11.4 in the Appendix shows that investment is increasing in unsecured debt issuance when the dividend constraint binds.

Unlike before, it is possible that shareholders may wish to buy back debt when a dividend constraint is in place. Consequently, deviations of unsecured debt management from an unrestricted benchmark can be thought as having two drivers:

$$b^{\text{rest}} - b^{\text{unrest}} = -\underbrace{l \frac{\theta c^u}{j_{ff}}}_{\text{Reduced issuance motive}} - \underbrace{l \frac{\lambda \bar{p}}{j_{ff}}}_{\text{Debt repurchase motive}}$$

Interpreting l as the value shareholders assign to relaxing the dividend constraint, the more value shareholders assign to relaxing the constraint (and thereby being able to payout dividends), the more intense is their motivation to reduce issuance and to even buyback debt. This decision can be pinned down by the following inequality:

$$(1-l)\frac{\theta c^u}{j_{ff}} < l\frac{\lambda \bar{p}}{j_{ff}}$$
$$\Rightarrow \theta c^u < \frac{l}{1-l}\lambda \bar{p}$$

That is, when the debt tax shield is below some threshold defined by shadow value of issuing dividends and the jump probability weighted pre-crisis debt price, the firm will find it beneficial to make net repurchases of unsecured debt.

Section 4.3 discusses the numerical results for models with dividend restrictions accompanying unsecured debt intervention, with and without debt repurchase restrictions. As mentioned, the motivation is to see if firms will redirect the proceeds from unsecured debt intervention to investment from payouts when dividend restrictions are in place. While this does occur, the value of equity also falls. Moreover, firms engage in net repurchases of unsecured debt and when repurchases are restricted, they choose not to issue at all. Hence, while investment and default dynamics improve when unsecured debt intervention is combined with dividend restrictions, the numerical analysis suggests that firms may not voluntarily participate in such a program.

4. Numerical Results

4.1. Model Calibration and Fit

Table 1 reports the calibrated parameters used in the numerical solution of the model reported in this section. The parameters are chosen to correspond with those used by Crouzet and Tourre (2021), with the exception of the curvature of the capital adjustment cost function, γ, the proportion of the capital stock pledgeable as collateral, α, and the initial distribution of debt to capital, f. Crouzet and Tourre (2021) estimate γ, the productivity of capital, A, and the volatility of capital quality, σ, targeting the slope of investment with respect to net debt to EBITDA, the average investment rate, and the average net debt to EBITDA as empirical moments for identification (see Table 2).

The specification for capital adjustment costs differs in this paper to ensure that investment is non-negative in equilibrium, which simplifies the modeling of fully collateralized secured debt. The selected value for γ lies in the range reported by

Table 1. Calibrated Parameters

Parameter	Description	Value	Source
γ	Adjustment Cost Curvature	16	Caballero and Engel (1999)
A	Capital Productivity	0.24	Crouzet and Tourre (2021)
α	Capital Stock Pledgeability	0.20	Catherine et al. (2022)
σ	Capital Quality Volatility	0.31	Crouzet and Tourre (2021)
θ	Corporate Income Tax Rate	0.35	OECD (2020)
m^u, m^s	Debt Amortization Rate	0.10	Saretto and Tookes (2013)
δ	Depreciation Rate	0.10	Hennessy and Whited (2005)
r	Risk-Free Rate	0.05	Crouzet and Eberly (2020)

The table reports the calibrated parameters used in the numerical solution of the model presented in Section 4. The parameters are chosen to correspond with those used by Crouzet and Tourre (2021), with the exception of the curvature of the capital adjustment cost function, γ, the proportion of the capital stock pledgeable as collateral, α, and the initial distribution of debt to capital, f. Crouzet and Tourre (2021) estimate γ, the productivity of capital, A, and the volatility of capital quality, σ, targeting the slope of investment with respect to (net) debt to EBITDA, the average investment rate, and the average (net) debt to EBITDA as empirical moments for identification (see Table 2). The capital adjustment costs differ in this paper to ensure that investment is non-negative in equilibrium, which simplifies the modeling of fully collateralized secured debt. The selected value for γ lies in the range reported by Falato et al. (2022) of 2 to 20, matching the value of 16 estimated by Caballero and Engel (1999). The selected value for α corresponds to the lower-bound of the range (0.20 to 0.25) estimated by Catherine et al. (2022). This parameter is not present in Crouzet and Tourre (2021), since they do not model secured debt. The initial distribution of debt to capital is shown in Figure A1. It is proxied by gross liabilities to assets taken from Compustat.

Falato et al. (2022) of 2 to 20, matching the value of 16 estimated by Caballero and Engel (1999). The selected value for α corresponds to the lower-bound of the range (0.20 to 0.25) estimated by Catherine et al. (2022). This parameter is not present in Crouzet and Tourre (2021), since they do not model secured debt.

The initial distribution of debt to capital is shown in Figure A1. It is proxied by gross liabilities to assets taken from Compustat. The choice of initial distribution does not impact the model solution but does affect the calculation of the model implied moments. While Crouzet and Tourre (2021) target debt net of cash holdings for empirical moments, initializing the model with the empirical distribution of net debt to capital would lead to negative values for the state variable, which is ruled out by assumption and presents complications (i.e. net lending by non-financial firms) that are out of scope for this paper.

Table 2 reports the implied moments from the model presented in this paper (column corresponding to 'Model') and compares these against the empirical moments computed by Crouzet and Tourre (2021) ('Data'), as well as the moments implied by their model ('CT21'). Since Crouzet and Tourre (2021) target the average net debt to EBITDA, the average investment rate, and the slope of investment with respect to net debt to EBITDA in their estimation of parameters, these moments are particularly well-matched for their model. Since this paper initializes the distribution of debt to capital with the empirical gross leverage ratio, instead of net debt, the model implied

Table 2. Model Fit

Description	Source	Data	CT21	Model
Average Credit Spreads	Feldhütter and Schaefer (2018)	0.87-4.17	4.98	4.76
Average Debt Issuance Rate	Compustat	25.7	17.9	23.1
Average Debt to EBITDA	Compustat	2.13	2.14	3.08
Average Equity Payout Rate	Compustat	4.6	3.0	5.0
Average Inverse Interest Coverage Ratio	Compustat	11.3	10.7	15.4
Average Investment Rate	Compustat	11.28	11.28	9.55-16.88
Default Rate (2019)	S&P (2025)	1.3	1.5	1.3
Slope of Inv. wrt Debt to EBITDA	Compustat	-1.04	-1.04	-1.15

The table reports the implied moments from the model presented in this paper (column corresponding to 'Model') and compares these against the empirical moments computed by Crouzet and Tourre (2021) ('Data'), as well as the moments implied by their model ('CT21'). Crouzet and Tourre (2021) estimate γ, the productivity of capital, A, and the volatility of capital quality, σ, reported in Table 1, targeting the slope of investment with respect to debt-to-EBITDA, the average investment rate, and the average debt-to-EBITDA as empirical moments for identification. As a result, the 'Data' and 'CT21' moments closely match for these variables. Crouzet and Tourre (2021) reports empirical moment for debt to EBITDA for debt net of cash. In contrast, this paper uses the initial debt to capital distribution for gross book leverage, to avoid negative net debt values (see Figure A1). This results in higher model implied moments for debt to EBITDA and inverse interest coverage ratio. The default rate comes from S&P (2025) data for 2019. While the values are similar, the model implied default rate is lower for this paper, despite higher starting debt levels. This can be due to two forces: 1. firm restructuring, rather than liquidation, after default, in Crouzet and Tourre (2021) and 2. secured debt enhancing overall debt capacity in this paper. Similarly, the model implied average credit spreads are slightly lower in this paper than in Crouzet and Tourre (2021). The empirical moments for average credit spreads come from Feldhütter and Schaefer (2018) and cover the range of credit spreads for investment-grade (IG) and high-yield (HY) firms. Crouzet and Tourre (2021) present investment rates gross of capital adjustment costs. The range of investment rates with and without capital adjustment costs are reported for this paper; the interval contains both the empirical and model moments reported by Crouzet and Tourre (2021). Higher gross investment rates in this paper are driven by the presence of secured debt which incentivizes higher investment to generate more pledgeable collateral. The moments for average debt issuance, average equity payout rate, and the slope of investment with respect to debt to EBITDA are broadly aligned across the three estimates.

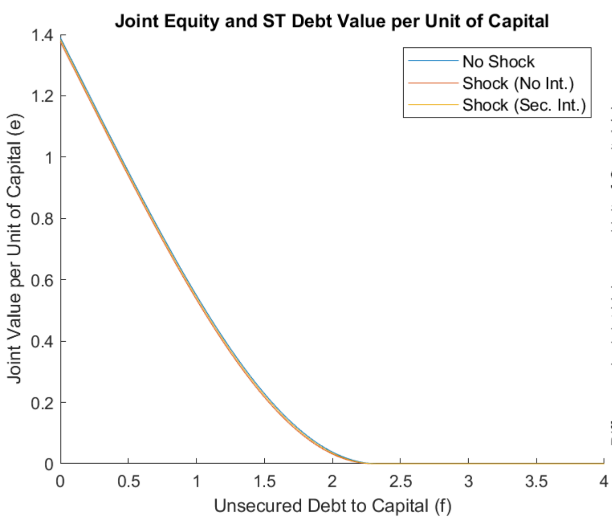
moments are larger for debt to EBITDA and inverse interest coverage ratio.

The default rate comes from S&P (2025) data for 2019. While the values are similar, the model implied default rate is lower for this paper, despite higher starting debt levels. This is consistent with secured debt enhancing overall debt capacity. Similarly, the model implied average credit spreads are slightly lower in this paper than in Crouzet and Tourre (2021). The empirical moments for average credit spreads come from Feldhütter and Schaefer (2018) and cover the range of credit spreads for investment-grade (IG) and high-yield (HY) firms.

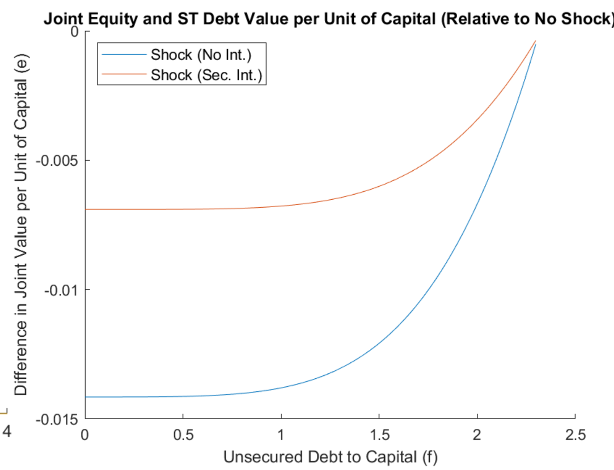
Crouzet and Tourre (2021) present investment rates gross of capital adjustment costs. The range of investment rates with and without capital adjustment costs are reported for this paper; the interval contains both the empirical and model moments reported by Crouzet and Tourre (2021). Higher gross investment rates in this paper are driven by the presence of secured debt which incentivizes higher investment to generate more pledgeable collateral. The moments for average debt issuance, average equity payout rate, and the slope of investment with respect to debt to EBITDA are broadly aligned across the three estimates.

4.2. Numerically Solved Model and Dynamics

Section 6.13 in the Appendix presents the details for numerically solving equity and unsecured debt prices in the model with short-term debt, as well as the model with dividend restrictions. Additional parameters are reported in Section 6.14.2 in the Appendix across the different economic environments considered for the model dynamics. As a preview of the results, secured debt intervention leads to higher equity values and investment, relative to no intervention. In addition, it leads to more favorable firm dynamics with longer-term improvements in investment and lower default rates. In contrast, while unsecured debt intervention does not alter current equity values and investment, it does lead to worse longer-term outcomes, driven by higher leverage and default rates among firms.



A. Levels



B. Relative to No Shock

Figure 1. Secured Debt Intervention Boosts Equity Prices After Shock

I estimate both current prices and investment policies, as well as the dynamics. Figure 1 shows the joint value of equity and short-term debt for the different economies considered in the baselines analysis: prices with no shock, with shock but no secured debt intervention, and with shock and secured debt intervention. Prices are unaffected by trading or unsecured debt interventions. That is, the trade and no trade solutions for the joint value of equity and short-term debt coincides. Consequently, secured debt intervention improves the joint value of equity and short-term debt relative to both no intervention and unsecured debt intervention, as emphasized in Figure 1B.

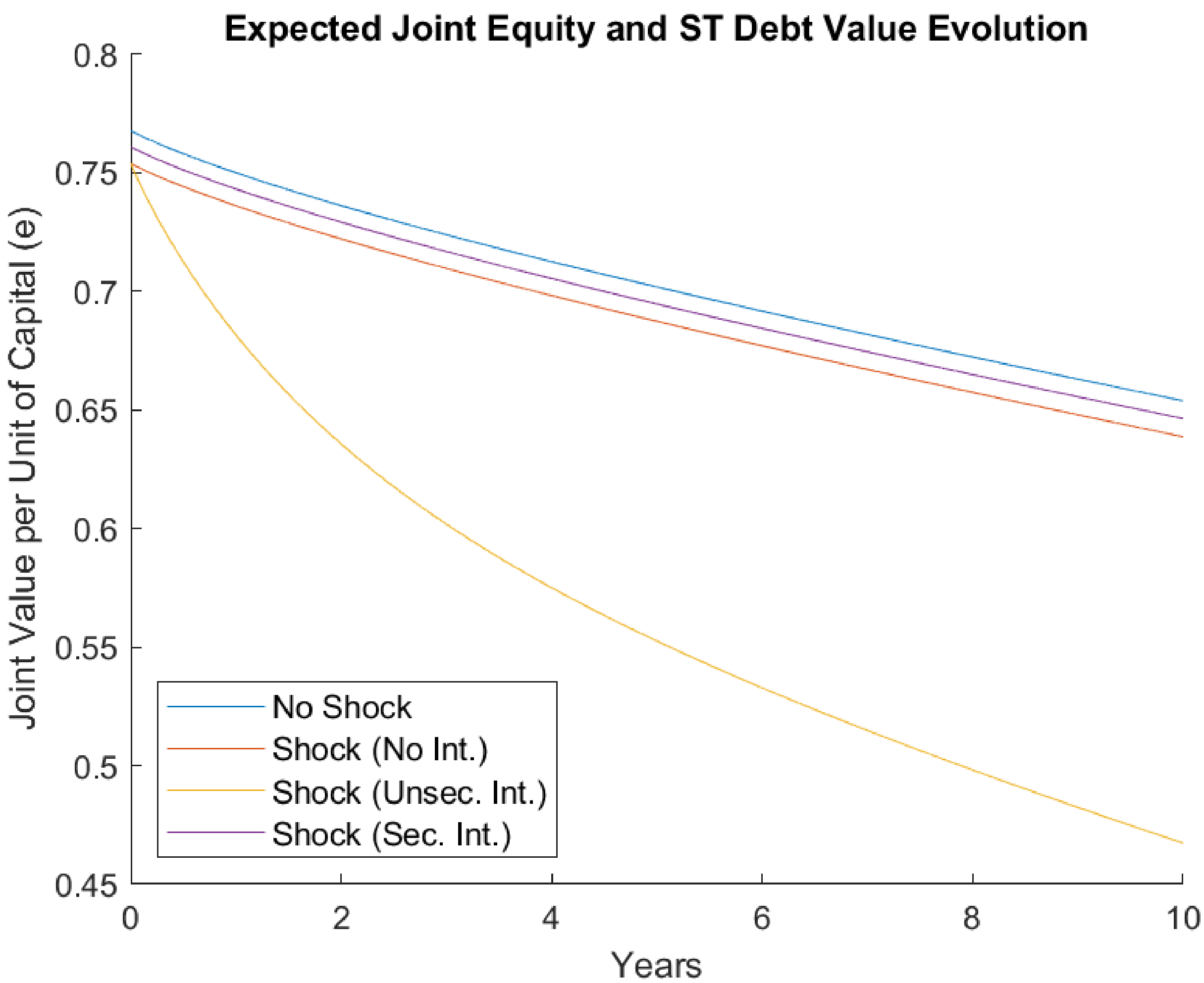
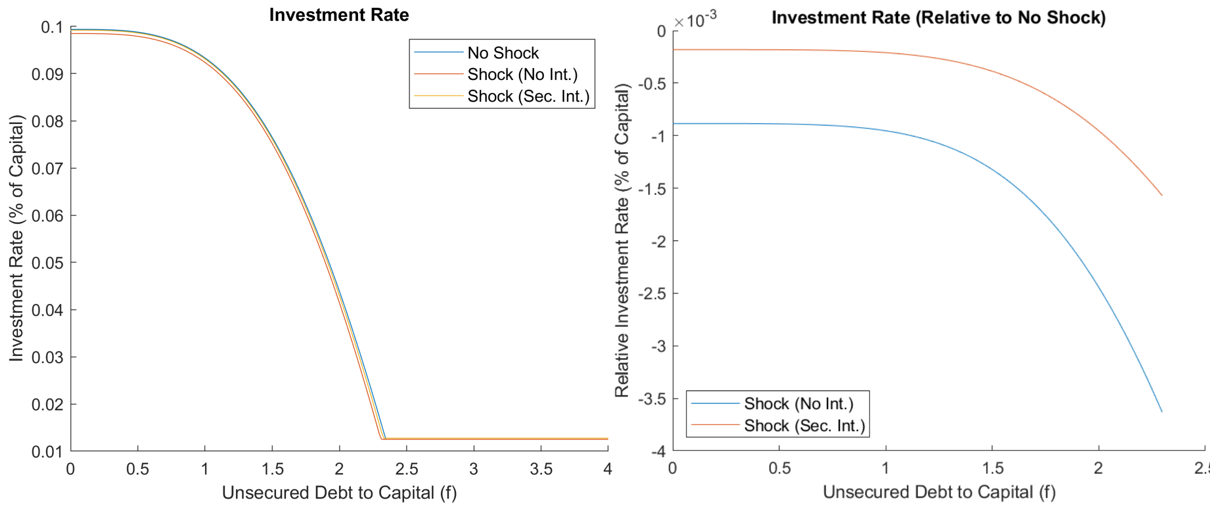


Figure 2. Expected Equity Value Evolution Higher with Secured Debt Intervention

Figure 2 shows the long-run dynamics in average expected joint value of equity and short-term debt. The evolution of joint values shows that the economy with secured debt intervention dominates the economies with no intervention or unsecured debt intervention. In fact, the evolution of expected joint values are worst with unsecured debt interventions, reflecting the perverse effects induced by no commitment: unsecured debt intervention boosts dividend payments exactly at the same rate as it accelerates the firms movement toward default.



A. Levels

B. Relative to No Shock

Figure 3. Investment Rate Higher with Secured Debt Intervention

Consistent with Figure 3, we see that secured debt intervention boosts investment relative to no intervention (and hence, unsecured debt intervention) after an economy experiences a shock.

The evolution of expected investment rates, shown in Figure 4, is analogous to that of equity prices, with secured debt intervention resulting in a higher path of expected investment versus no intervention and, especially, unsecured debt intervention.

Unsecured debt prices are shown in Figure 5. In contrast to joint equity and short-term debt prices, dispersion in unsecured debt prices is initially low for lower leverage and increases closer to the default threshold. Given the parameter values used, debt prices are concave and lead to monotonically decreasing issuance rates, as seen in Figure 6A. Following the shock, unsecured debt issuance falls for the cases with no intervention and secured debt intervention but rises with unsecured debt intervention, as seen in Figure 6B.

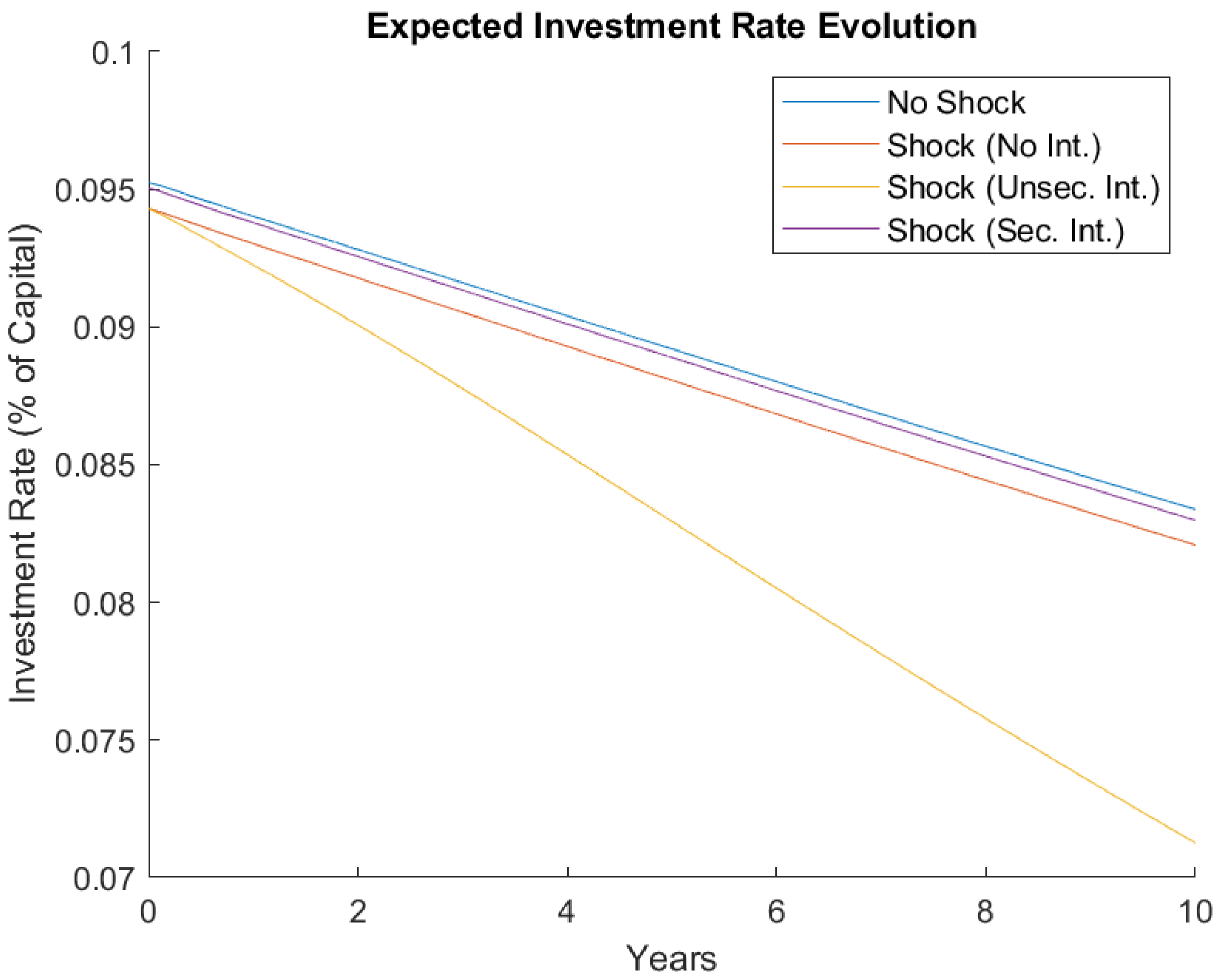
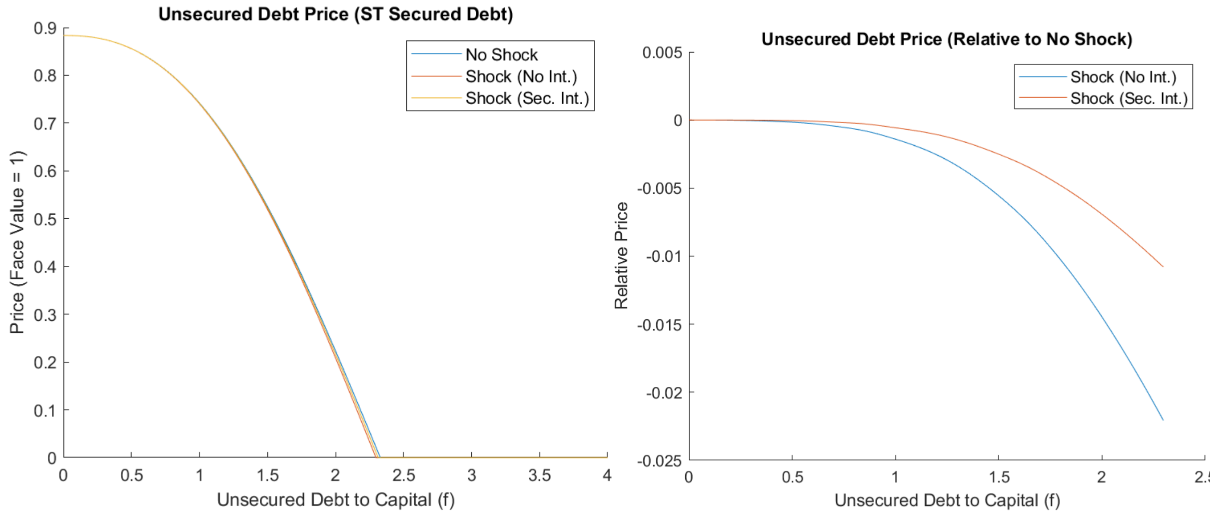
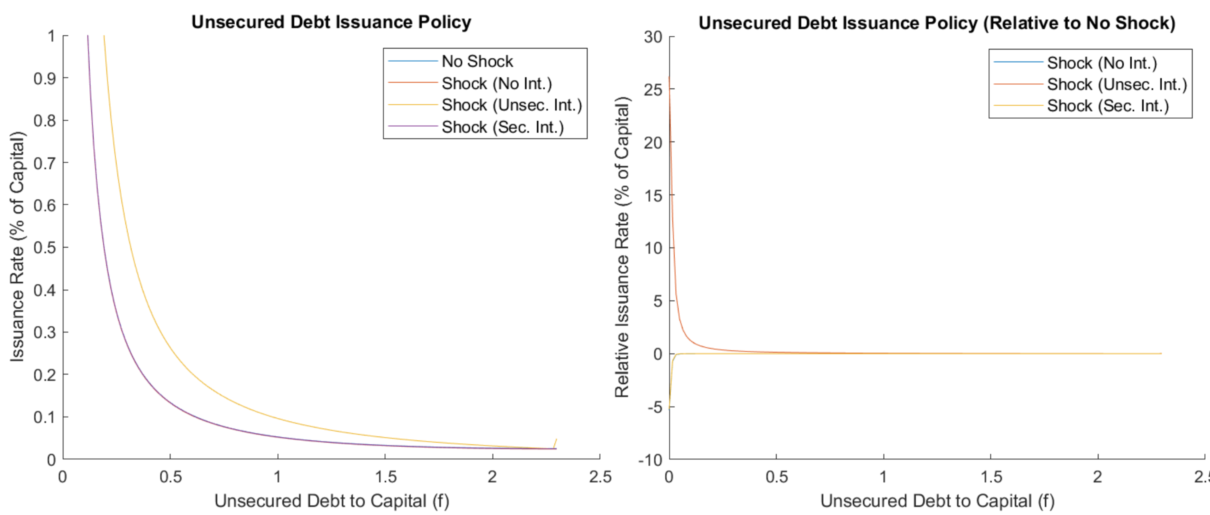


Figure 4. Evolution of of Expected Investment Rates Higher with Secured Debt Intervention



A. Levels B. Relative to No Shock

Figure 5. Unsecured Debt Prices Higher With Secured Debt Intervention for More Leveraged Firms



A. Levels B. Relative to No Shock

Figure 6. Unsecured Debt Issuance Rate Higher with Intervention

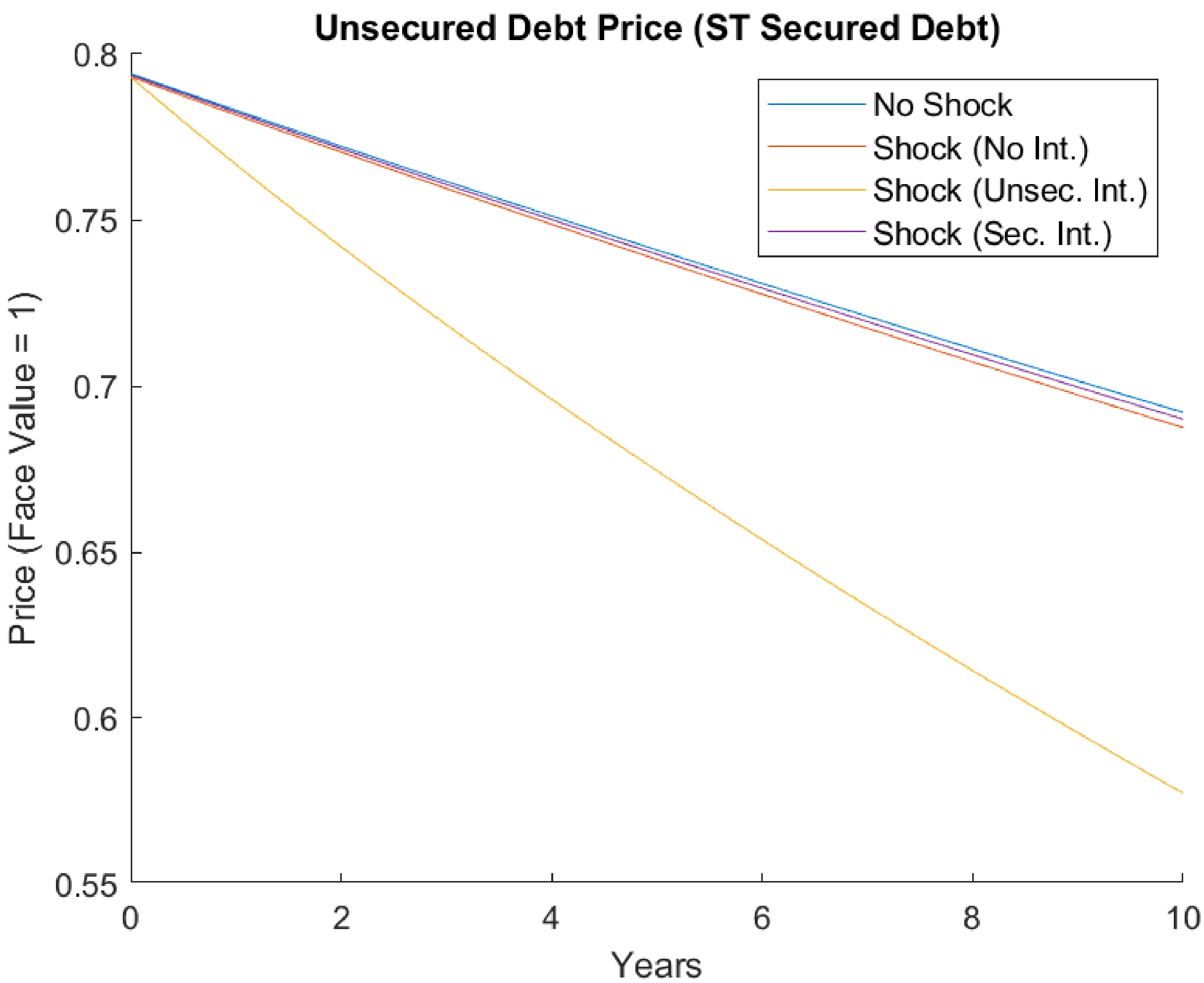
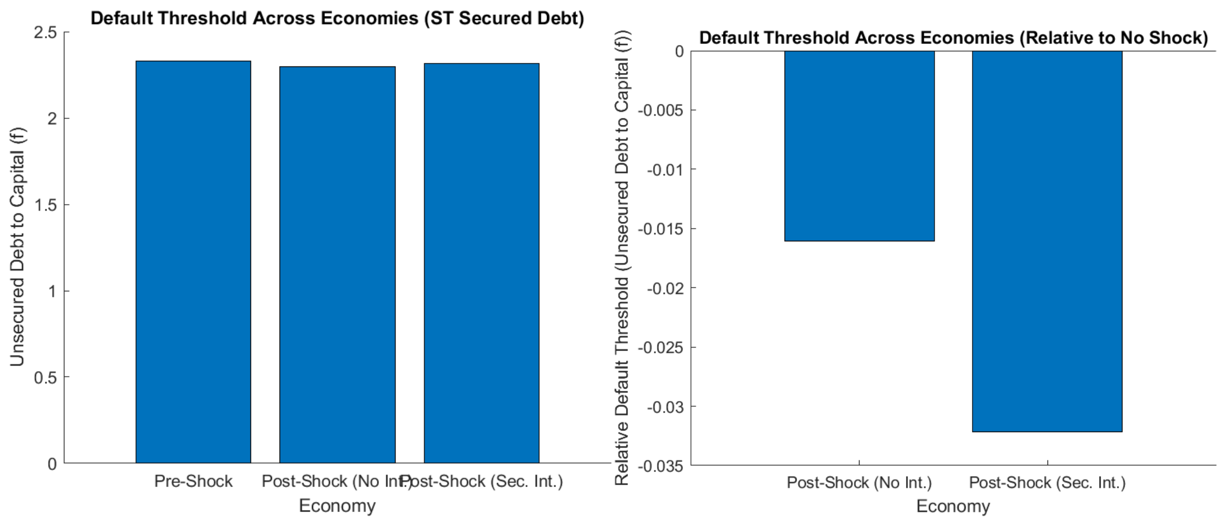


Figure 7. Secured Debt Intervention Boosts Unsecured Debt Price More Than Unsecured Debt Intervention

As with equity prices, the evolution of unsecured debt prices is higher with secured debt intervention than unsecured debt intervention, as seen in Figure 7. In fact, the economy with unsecured debt intervention has lower future expected prices. This underscores how unsecured debt intervention may be counter-productive: firms increase debt issuance in response to the government becoming the marginal buyer at lower discount rates, which increases in default risk. In current valuations, these two forces exactly offset each other, such that the unsecured debt price with unsecured debt intervention is unchanged from the no trade unsecured debt price.



A. Levels B. Relative to No Shock

Figure 8. Secured Debt Intervention Pushes Back Default Threshold

In addition to improving prices and investment, secured debt intervention also increases the debt tolerance of firms, as shown in Figure 8. While the default threshold decreases post-shock, the decrease is lower with secured debt intervention than no intervention, as seen in in Figure 8B. This provides another benefit of secured debt intervention in addition to directly boosting investment.

Consistent with higher default thresholds, cumulative default is lower with secured debt intervention, as seen in Figure 9. Echoing the evolution of unsecured debt prices, unsecured debt intervention results in the greatest number of cumulative defaults, despite not affecting the default threshold. Instead, unsecured debt intervention accelerates the path to default by encouraging unsecured debt issuance.

Figure 10 depicts the long-run distribution of surviving firms. All of the distributions are right-skewed because of the mean reverting nature of the dynamics, and the proportionally slower rate of issuance for more indebted firms.

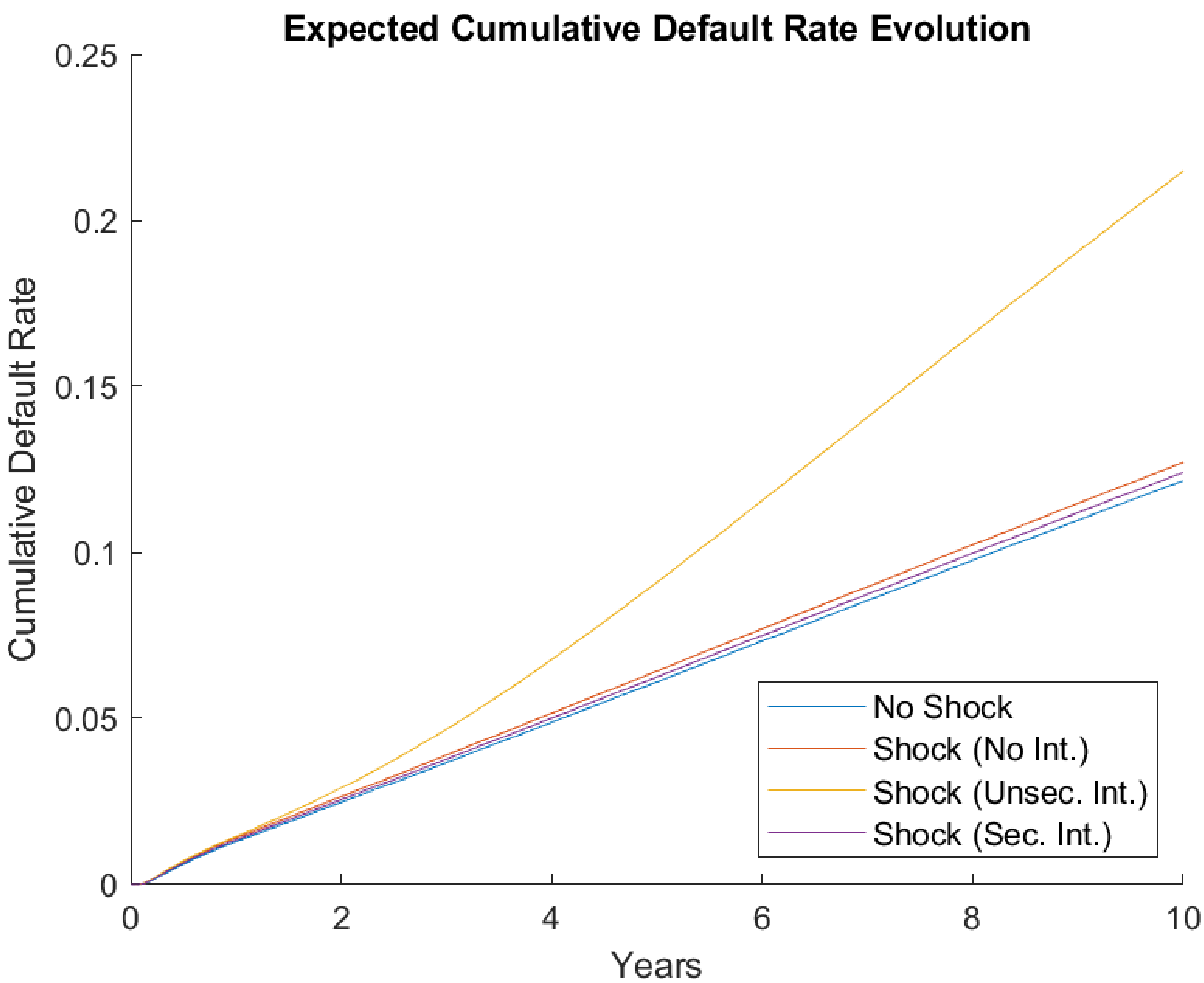
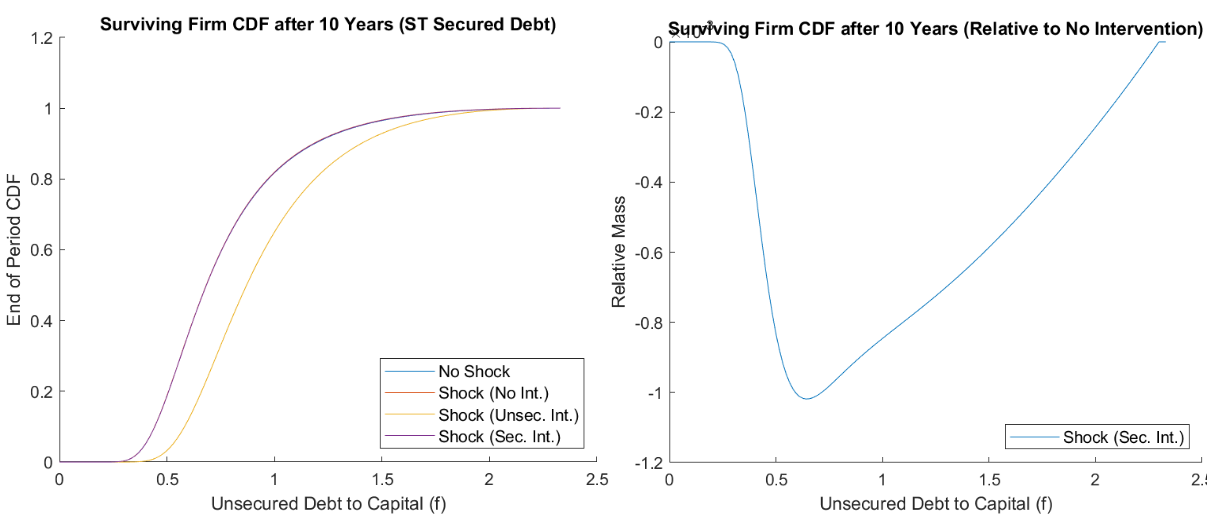


Figure 9. Secured Debt Intervention Reduces Expected Cumulative Default Rates

Figure 10. Secured Debt Intervention Supports Higher Leverage Ratios



A. Levels

B. Relative to No Intervention

Figure 11. Secured Debt Intervention Distribution Features First Order Stochastic Dominance Over Unsecured Debt Intervention

Figure 11 shows the CDFs of surviving firms. Unsecured debt intervention results in a notable rightward shift in the CDF of surviving firms, suggesting that these firms have higher levels of leverage. Figure 11B computes the difference between the CDF for the economy where there is secured debt intervention versus the economy with no intervention. The non-negative values indicate that the CDF for the economy with secured debt intervention exhibits (weak) first order stochastic dominance over the corresponding distribution for the economy with no intervention, as well as the economy with unsecured debt intervention.

4.3. Dividend and Debt Repurchase Restrictions with Unsecured Debt Intervention

Section 6.13.6 in the Appendix provides details on the numerical solution method to solve the extension to the baseline model with nonlinear policy constraints. I consider both dividend restrictions with and without a constraint on unsecured debt repurchases. Both economies with dividend restrictions also feature unsecured debt intervention (see Section 6.14.2 in the Appendix). Critically, the dividend restriction is only in place for the duration of the crisis; otherwise, if it were permanent, equity prices would fall to zero as shareholders would derive no value from owning equity.

Dividend restrictions lead to more beneficial investment outcomes, lower leverage, and more favorable investment and default dynamics. Unlike before, unsecured debt prices are also higher with intervention. However, they lead to lower equity prices and even create an incentive for firms to repurchase unsecured debt. If debt repurchases are further restricted, firms choose not to issue unsecured debt when they otherwise would have repurchased debt. All together, the numerical solutions suggest firms would not voluntarily participate in an unsecured debt intervention program with dividend restrictions.

5. Conclusion

Empirical research has shown that central bank corporate bond purchase programs in Europe and the United States led to an increase in leverage for directly targeted firms. The payouts of these firms to shareholders increased relative to other firms, while investment did not. A commonality of both programs is that they primarily involved interventions in the unsecured debt of financially unconstrained firms. This paper shows that a dynamic capital structure model where firms cannot commit to a debt issuance policy ex ante can reproduce these stylized facts.

Without commitment, firms accelerate the issuance of unsecured debt following intervention. The additional proceeds are distributed to shareholders and are not used for investment. Higher firm payouts and higher leverage directly offset each other to leave the firm's current discounted equity valuation unchanged. However, the greater debt burden translates into worse investment and default dynamics for firms. In contrast, secured debt issuance features implicit commitment induced by the firm's collateral constraint. While the scope of secured debt intervention is far more limited, I show theoretically and numerically that secured debt intervention can improve investment outcomes and default dynamics for firms, both relative to the case of no intervention and especially the case of unsecured debt intervention.

Imposing dividend restrictions while intervening in unsecured debt reduces the negative impact of the lack of firms' commitment to an ex ante debt policy, leading to higher unsecured debt prices, greater investment, and more favorable credit dynamics. However, dividend restrictions also lead to a drop in firms' equity valuation and actually induce firms to repurchase unsecured debt. Restricting debt repurchases further increases investment but does not improve equity valuation. These results suggest that firms would not voluntarily participate in unsecured debt intervention programs with dividend restrictions, since it would not be optimal from a valuation standpoint.

To the extent that central banks motivated their interventions by arguing real outcomes would be improved from loosening financial conditions, the findings in this paper on the improved dynamics generated by secured debt intervention, compared to unsecured debt intervention, are important. However, this paper abstracts away from any potential moral hazard induced by such credit programs. Nor does it address other concerns around directing monetary stimulus to relatively unconstrained and large firms, such as negative effects on competitiveness or welfare implications. Policymakers will need to balance these concerns against any potential benefits in future interventions in corporate credit markets.

References

Abel, Andrew B. 2016. "Investment with Leverage."

Abel, Andrew B. 2018. "Optimal Debt and Profitability in the Trade-Off Theory." *The Journal of Finance* 73 (1): 95–143.

Abidi, Nordine, and Ixart Miquel-Flores. 2018. "Who Benefits from the Corporate QE? A Regression Discontinuity Design Approach." *SSRN Electronic Journal*.

Acharya, Viral, and Guillaume Plantin. 2025. "Monetary Easing, Leveraged Payouts, and Lack of Investment." *Management Science*: mnsc.2022.01440.

Acharya, Viral V, and Sascha Steffen. 2020. "The Risk of Being a Fallen Angel and the Corporate Dash for Cash in the Midst of COVID." *The Review of Corporate Finance Studies* 9 (3): 430–471.

Almeida, Heitor, Murillo Campello, and Michael S. Weisbach. 2004. "The Cash Flow Sensitivity of Cash." *The Journal of Finance* 59 (4): 1777–1804.

Becker, Bo, and Efraim Benmelech. 2021. "The Resilience of the U.S. Corporate Bond Market During Financial Crises."Technical Report w28868, National Bureau of Economic Research. Cambridge, MA.

Benmelech, Efraim, Nitish Kumar, and Raghuram Rajan. 2022. "The secured credit premium and the issuance of secured debt." *Journal of Financial Economics* 146 (1): 143–171.

Benmelech, Efraim, Nitish Kumar, and Raghuram Rajan. 2024. "The Decline of Secured Debt." *The Journal of Finance* 79 (1): 35–93.

Boyarchenko, Nina, Anna Kovner, and Or Shachar. 2022. "It's what you say and what you buy: A holistic evaluation of the corporate credit facilities." *Journal of Financial Economics* 144 (3): 695–731.

Caballero, Ricardo J., and Eduardo M. R. A. Engel. 1999. "Explaining investment dynamics in U.S. manufacturing: A generalized (S,s) approach." *Econometrica* 67 (4): 783–826.

Catherine, Sylvain, Thomas Chaney, Zongbo Huang, David Sraer, and David Thesmar. 2022. "Quantifying Reduced-Form Evidence on Collateral Constraints." *The Journal of Finance* 77 (4): 2143–2181.

Crouzet, Nicolas, and Janice Eberly. 2020. "Rents and Intangibles: a Q+ Framework."

Crouzet, Nicolas, and Fabrice Tourre. 2021. "Can the cure kill the patient? Corporate credit interventions and debt overhang." *SSRN Electronic Journal*.

Darmouni, Olivier, and Kerry Y. Siani. 2024. "Bond market stimulus: Firm-level evidence." *Journal of Monetary Economics*: 103728.

De Santis, Roberto A., and Andrea Zaghini. 2021. "Unconventional monetary policy and corporate bond issuance." *European Economic Review* 135: 103727.

DeMarzo, Peter, Zhiguo He, and Fabrice Tourre. 2023. "Sovereign Debt Ratchets and Welfare Destruction." *Journal of Political Economy* 131 (10): 2825–2892.

Demarzo, Peter M. 2019. "Presidential Address: Collateral and Commitment." *The Journal of Finance* 74 (4): 1587–1619.

Demarzo, Peter M., and Zhiguo He. 2021. "Leverage Dynamics without Commitment." *The Journal of Finance* 76 (3): 1195–1250.

Denis, David J., and Valeriy Sibilkov. 2010. “Financial Constraints, Investment, and the Value of Cash Holdings.” *Review of Financial Studies* 23 (1): 247–269.

Dutordoir, Marie, Joshua Shemesh, Chris Veld, and Qing Wang. 2024. “Can existing corporate finance theories explain security offerings during the COVID-19 pandemic?” *Journal of Empirical Finance* 79: 101558.

D’Amico, Stefania, Vamsidhar Kurakula, and Stephen Lee. 2020. “Impacts of the Fed Corporate Credit Facilities through the Lenses of ETFs and CDX.”Technical report, Federal Reserve Bank of Chicago.

Elgouacem, Assia, and Riccardo Zago. 2023. “Share Buybacks, Monetary Policy and the Cost of Debt.” *International Journal of Central Banking* 19 (2): 295–349.

Falato, Antonio, Dalida Kadyrzhanova, Jae Sim, and Roberto Steri. 2022. “Rising Intangible Capital, Shrinking Debt Capacity, and the U.S. Corporate Savings Glut.” *The Journal of Finance* 77 (5): 2799–2852.

Faulkender, Michael, and Mitchell A Petersen. 2006. “Does the Source of Capital Affect Capital Structure?” *The Review of Financial Studies* 19 (1).

Feldhütter, Peter, and Stephen M. Schaefer. 2018. “The myth of the credit spread puzzle.” *The Review of Financial Studies* 31 (8): 2897–2942.

Flanagan, Thomas, and Amiyatosh Purnanandam. 2020. “Corporate Bond Purchases After COVID-19: Who Did the Fed Buy and How Did the Markets Respond?”.

Gilchrist, Simon, Bin Wei, Vivian Z Yue, and Egon Zakrajšek. 2021. “The Fed takes on corporate credit risk: an analysis of the efficacy of the SMCCF.”

Greenwald, Daniel L., John Krainer, and Pascal Paul. 2023. “The Credit Line Channel.” *Federal Reserve Bank of San Francisco, Working Paper Series*: 1.000–96.000.

Grosse-Rueschkamp, Benjamin, Sascha Steffen, and Daniel Streitz. 2019. “A capital structure channel of monetary policy.” *Journal of Financial Economics* 133 (2): 357–378.

Haddad, Valentin, Alan Moreira, and Tyler Muir. 2021. “When Selling Becomes Viral: Disruptions in Debt Markets in the COVID-19 Crisis and the Fed’s Response.” *The Review of Financial Studies* 34 (11): 5309–5351.

Halling, Michael, Jin Yu, and Josef Zechner. 2020. “How Did COVID-19 Affect Firms’ Access to Public Capital Markets?” *The Review of Corporate Finance Studies* 9 (3): 501–533.

Harford, Jarrad, and Vahap B. Uysal. 2014. “Bond market access and investment.” *Journal of Financial Economics* 112 (2): 147–163.

Hayashi, Fumio. 1982. “Tobin’s Marginal q and Average q: A Neoclassical Interpretation.” *Econometrica* 50 (1): 213.

Hennessy, Christopher A., and Toni M. Whited. 2005. “Debt dynamics.” *The Journal of Finance* 60 (3): 1129–1165.

Hotchkiss, Edith, Greg Nini, and David C Smith. 2022. “The Role of External Capital in Funding Cash Flow Shocks: Evidence From the COVID-19 Pandemic.”

Hu, Yunzhi, Felipe Varas, and Chao Ying. 2024. “Debt Maturity Management.”

Kargar, Mahyar, Benjamin Lester, David Lindsay, Shuo Liu, Pierre-Olivier Weill, and Diego Zúñiga. 2021. “Corporate Bond Liquidity during the COVID-19 Crisis.” *The Review of Financial Studies* 34 (11): 5352–5401.

Kiyotaki, Nobuhiro, and John Moore. 1997. "Credit Cycles." *Journal of Political Economy* 105 (2): 211–248.
Leland, Hayne E. 1994. "Corporate Debt Value, Bond Covenants, and Optimal Capital Structure." *The Journal of Finance* 49 (4): 1213–1252.
Li, Wenhao, and Ye Li. 2024. "Firm Quality Dynamics and the Slippery Slope of Credit Intervention." *SSRN Electronic Journal*.
Momin, Rayhan. 2025. "Heterogeneous Treatment Effects and Counterfactual Policy Targeting Using Deep Neural Networks: An Application to Central Bank Corporate Credit Facilities."
Momin, Rayhan, and Jessica S. Li. 2022. "The Causal Effect of the Fed's Corporate Credit Facilities on Eligible Issuer Bonds." *SSRN Electronic Journal*.
OECD. 2020. "OECD Tax Database."
O'Hara, Maureen, and Xing (Alex) Zhou. 2021. "Anatomy of a liquidity crisis: Corporate bonds in the COVID-19 crisis." *Journal of Financial Economics* 142 (1): 46–68.
Pazarbasi, Altan. 2025. "Cash Heterogeneity and the Payout Channel of Monetary Policy."
Pegoraro, Stefano, and Mattia Montagna. 2025. "Demand-Driven Bond Financing in the Euro Area." *Journal of Financial and Quantitative Analysis*: 1–52.
Rampini, Adriano A., and S. Viswanathan. 2010. "Collateral, Risk Management, and the Distribution of Debt Capacity." *The Journal of Finance* 65 (6): 2293–2322.
Saretto, Alessio, and Heather E. Tookes. 2013. "Corporate leverage, debt maturity, and credit supply: The role of credit default swaps." *The Review of Financial Studies* 26 (5): 1190–1247.
S&P. 2025. "Default, Transition, and Recovery: 2024 Annual Global Corporate Default And Rating Transition Study."
Todorov, Karamfil. 2020. "Quantify the quantitative easing: Impact on bonds and corporate debt issuance." *Journal of Financial Economics* 135 (2): 340–358.
Whited, Toni M. 1992. "Debt, Liquidity Constraints, and Corporate Investment: Evidence from Panel Data." *The Journal of Finance* 47 (4): 1425–1460.
Zaghini, Andrea. 2019. "The CSPP at work: Yield heterogeneity and the portfolio rebalancing channel." *Journal of Corporate Finance* 56: 282–297.

Appendix 6. Appendix

6.1. Strict Convexity of the Joint Value Function

PROPOSITION A1 (Strict Convexity of the Joint Value Function). *Let $j(f)$ denote the maximized joint value function of the firm given a debt to capital ratio f. Then, $j(f)$ is strictly convex in the continuation region (outside of default); that is, for any two feasible leverage levels f_1 and f_2 and for any $\lambda \in (0,1)$, if we define*

$$f_\lambda = \lambda f_1 + (1-\lambda) f_2,$$

we have

$$\lambda j(f_1) + (1-\lambda) j(f_2) > j(f_\lambda).$$

Proof. Let the firm's optimized joint value function be given by $j(f)$ with values $j(f_1)$ and $j(f_2)$ for feasible leverage valeus f_1 and f_2. Consider the convex combination

$$f_\lambda = \lambda f_1 + (1-\lambda) f_2,$$

which is a feasible debt level by the continuity of j.

Since $j(f_1)$ and $j(f_2)$ represent optimized values, any deviation from the optimal policies cannot produce a higher value. That is, if the firm with leverage f_1 deviates to f_λ, then

$$j(f_1) > j(f_\lambda) + \left(f_\lambda - f_1\right) p(f_\lambda),$$

where $p(f_\lambda)$ is the price of unsecured debt given leverage f_λ. Since f_λ lies in the continuation region, $p(f_\lambda) > 0$. Then, $(f_\lambda - f_1)\, p(f_\lambda)$ is the incremental proceeds from deviating to f_λ from f_1. Analogously, a deviation from f_2 gives

$$j(f_2) > j(f_\lambda) + \left(f_\lambda - f_2\right) p(f_\lambda).$$

Take the weighted average of these two inequalities by multiplying the first by λ and the second by $(1-\lambda)$. Add the two together to obtain:

$$\lambda j(f_1) + (1-\lambda) j(f_2) > \lambda j(f_\lambda) + \lambda(f_\lambda - f_1)\, p(f_\lambda) + (1-\lambda) j(f_\lambda) + (1-\lambda)(f_\lambda - f_2)\, p(f_\lambda).$$

This simplifies to:

$$\lambda j(f_1) + (1-\lambda) j(f_2) > j(f_\lambda) + \left[\lambda(f_\lambda - f_1) + (1-\lambda)(f_\lambda - f_2)\right]\, p(f_\lambda).$$

Note that by the definition of f_λ,

$$\lambda(f_\lambda - f_1) + (1-\lambda)(f_\lambda - f_2) = 0.$$

Thus, the inequality reduces to:

$$\lambda j(f_1) + (1-\lambda) j(f_2) > j(f_\lambda).$$

Since the above inequality holds for any $\lambda \in [0,1]$ and any two feasible leverage levels f_1 and f_2, it follows by definition that the function $j(f)$ is strictly convex in the continuation region (outside of default). □

6.2. Collateral Constraint with Short-Term Secured Debt

This section provides the proof for Proposition 1.

Proof. Note that by the complementary slackness condition:

$$-l^s(s-\alpha) = 0, l^s \geq 0$$

That is, either the constraint binds, $l^s > 0, s = \alpha$, or there is slack, $l^s = 0, s < \alpha$.

To obtain equilibrium conditions on the derivatives of the value function, take the first order condition with respect to b^s and take successive derivatives to obtain:

$$\begin{aligned} j_s &= -1 \\ j_{ss} &= 0 \\ j_{sss} &= 0 \\ j_{ffs} &= 0 \\ j_{fs} &= 0 \end{aligned}$$

To obtain the value for p_s, take the first order condition of the HJB with respect to b^u, which yields $p = -j_f$. Differentiating with respect to s and using the expression for $j_{fs} = 0$ yields $p_s = 0$. Intuitively, this follows because the issuance of secured short-term debt does not entail additional bankruptcy costs.[19]

[19] This contrasts with Hu, Varas, and Ying (2024) where unsecured short-term debt is exposed to default risk via jumps, and the firm has the ability to issue risky short-term debt.

To obtain an expression for the Lagrange multiplier, l^s, differentiate the HJB equation with respect to s, appealing to the envelope condition for the controls and using the equilibrium derivative conditions for j:

$$\begin{aligned} 0 &= -(r-g+\delta)\, j_s - l^s + \theta c^s - c^s - (g-\delta)e_s \\ &= (r-g+\delta) - l^s + \theta c^s - c^s + (g-\delta) \\ \Rightarrow l^s &= r - c^s + \theta c^s \end{aligned}$$

Since $c^s = r$, $l^s > 0$, and the collateral constraint always binds.[20]

$$l^s = \theta c^s$$

Interpreting l^s as the value of the collateral constraint, it is intuitively equal to the debt tax shield when the coupon is set equal to the discount rate and the discount rate when the coupon equals zero.

With $s = \alpha$, and given the equilibrium derivative conditions, obtain:

$$\begin{aligned} 0 = \max_{g, b^u} \Big\{ &-(r-g+\delta)\, j + A - \theta(A - c^u f - c^s\alpha) - \Phi(g) - (c^u + m^u) f + p b^u - c^s \alpha + \alpha(g-\delta) \\ &+ [b^u - (g-\delta+m^u) f]\, j_f + \frac{1}{2}\sigma^2 f^2 j_{ff} \Big\} \end{aligned}$$

Appendix 6.4 shows that this is the same HJB equation that is obtained if one starts by assuming that the collateral constraint always binds before deriving the HJB equation.

□

6.3. Collateral Constraint with Long-Term Secured Debt

As before, shareholders and creditors are risk-neutral with discount rate r. Shareholders have the option to default on any security and at default, unsecured creditors have no recovery value while secured creditors receive the value of collateral. Equity investors are deep-pocketed. Capital adjustment costs are parameterized so that investment is nonnegative.

[20] Alternatively, if markets are segmented and lenders discount at a lower rate than borrowers, such that $c^s = \rho > r$, the collateral constraint still binds without the debt tax shield, i.e. $\theta = 0$

Firms' production technology, in revenue per unit of time, is given by:

$$Y_t = A_t K_t$$

The productivity parameter A_t evolves as:

$$\frac{dA_t}{A_t} = \mu dt + \sigma dZ_t$$

where μ is the constant rate of drift and dZ_t is the increment of a Brownian motion with distribution given by $dZ_t \sim \mathrm{N}(0, dt)$.

Capital K_t evolves as:

$$\frac{dK_t}{K_t} = (g_t - \delta)dt$$

where g_t is the endogenous investment rate (the capital stock rate of growth) and δ is the capital depreciation rate. The price of capital is fixed at 1. The absence of shocks to the capital process renders it locally deterministic. This helps to simplify the pricing of long-term secured debt.

Investing entails paying capital adjustment costs (per unit of capital), which are increasing and convex:

$$\Phi(g_t) = \frac{1}{2}\gamma g^2$$

where $\gamma > 0$ drives the cost of adjustment.

Unsecured debt has aggregate face value F_t and is endogenous. It matures at a Poisson rate m^u and hence, has expected maturity $1/m^u$. Individual bonds have face value equal to 1 and pay coupon rate $c^u = r$. Given default risk, unsecured debt is risky and pays $p_t < 1$. The evolution of unsecured debt stock is given by:

$$dF_t = \underbrace{-m^u F_t dt}_{\text{maturing debt}} + \underbrace{d\Gamma_t^u}_{\text{active debt management}}$$

I restrict my attention to the 'smooth' equilibrium where $d\Gamma_t^u = B_t^u dt$.

Firms can issue secured debt maturity at the rate $m^s = \delta$ and coupon rate equal to $c^s = r$. The collateral constraint is $S_t \leq \alpha K_t$, where S_t is the value of secured debt and α is the proportion of the capital stock that can be pledged. Since the investment rate

g_t will be non-negative (in equilibrium), the assumption that secured debt matures at the same rate as capital depreciates, no shocks to the capital stock (so, capital is locally deterministic), and because α is constant, secured debt will always be exactly collateralized. Hence, with $c^s = r$, secured debt is risk-free and issued at par equal to 1 for face value equal to 1. The secured debt stock evolves as:

$$dS_t = \underbrace{-m^s S_t dt}_{\text{maturing debt}} + \underbrace{B_t^s dt}_{\text{active debt management}}$$

Given constant corporate tax rate θ, firm pays $\theta(Y_t - c^u F_t - c^s S_t)$ in corporate taxes. Equity payout ($Payout_t$) equals:

$$\begin{aligned} &\underbrace{A_t K_t}_{\text{cash flows}} - \underbrace{\theta(A_t K_t - c^u F_t - c^s S_t)}_{\text{corporate taxes}} - \underbrace{\Phi(g_t) K_t}_{\text{investment cost}} \\ &- \underbrace{(c^u + m^u) F_t}_{\text{unsecured debt interest \& principal}} - \underbrace{(c^s + m^s) S_t}_{\text{secured debt interest \& principal}} \\ &+ \underbrace{p_t B_t^u}_{\text{unsecured debt issuance/repurchase}} + \underbrace{B_t^s}_{\text{secured debt issuance/repurchase}} \end{aligned}$$

If positive, equity pays out dividends to investors. If negative, equity receives a cash infusion from deep pocketed investors.

Shareholders take debt price p_t as given and maximize:

$$E(A, K, F, S) = \max_{g, B^u, B^s, \tau} \left\{ \mathbb{E}_0 \left[\int_0^\tau \exp(-rt)(Payout_t) dt \middle| A_0 = A, K_0 = K, F_0 = F, S_0 = S \right] \right\}$$

$$\frac{dA_t}{A_t} = \mu dt + \sigma dZ_t$$

$$\frac{dK_t}{K_t} = (g_t - \delta) dt$$

$$dF_t = (-m^u F_t + B_t^u) dt$$

$$dS_t = (-m^s S_t + B_t^s) dt$$

$$S_t \leq \alpha K_t$$

where τ is equity's endogenous default time.

Observe that capital K_t satisfies:

$$K_t = K_0 \exp\left(\int_0^t (g(s) - \delta) ds\right)$$

Then, we can show that the equity valuation equation is homogeneous of degree 1 in capital:

$$\begin{aligned} E(A, K, F, S) &= \max_{g, B^u, B^s, \tau} \left\{ \mathbb{E}_0 \left[\int_0^\tau \exp(-rt)(Payout_t) dt \middle| A_0 = A, K_0 = K, F_0 = F \right] \right\} \\ &= K \max_{g, B^u, B^s, \tau} \left\{ \mathbb{E}_0 \left[\int_0^\tau \exp\left(- \int_0^t (r - g_u + \delta) du \right) (Payout_t / K_t) dt \middle| A_0 = A, f_0 = f, s_0 = 0 \right] \right\} \\ &= K e(A, f, s) \end{aligned}$$

where,

$$\begin{aligned} f_t &\equiv F_t / K_t \\ s_t &\equiv S_t / K_t \\ b_t^u &\equiv B_t^u / K_t \\ b_t^s &\equiv B_t^s / K_t \end{aligned}$$

By applying Ito's lemma to the new state variables f and s, obtain:

$$\begin{aligned} df_t &= (b_t^u - (g_t + m^u - \delta) f_t) dt \\ ds_t &= (b_t^s - (g_t + m^s - \delta) s_t) dt \end{aligned}$$

Equity's HJB in the continuation region is given by:

$$\begin{aligned} 0 = \max_{g, b^u, b^s} \Big\{ & - (r - g + \delta) e(A, f, s) - l(s - \alpha) \\ & + A - \theta(A - c^u f - c^s s) - \Phi(g) - (c^u + m^u) f - (c^s + m^s) s + p(A, f, s) b^u + b^s \\ & + \mu A e_A(A, f, s) + \frac{1}{2} \sigma^2 A^2 e_{AA}(A, f, s) \\ & + \left[b^u - (g + m^u - \delta) f \right] e_f(A, f, s) \end{aligned}$$

$$+ \left[b^s - (g + m^s - \delta)s\right] e_s(A, f, s)\Big\}$$

where l is the Lagrange multiplier on the collateral constraint.

Taking the first order condition with respect to b^s to obtain e_s and take additional derivatives, as before.

$$\begin{aligned} e_s &= -1 \\ e_{sA} &= 0 \\ e_{sAA} &= 0 \\ e_{fs} &= 0 \\ e_{ss} &= 0 \end{aligned}$$

To obtain the value for p_s, differentiate the equity HJB with respect to b^u and obtain $p = -e_f$. Differentiate again with respect to s and use $e_{fs} = 0$ to obtain $p_s = 0$. As with secured short-term debt, the impact on unsecured debt from the issuance of secured debt is zero because secured debt is risk-free and its issuance does not incur additional bankruptcy costs.

Differentiate the HJB equation with respect to s, using the envelope condition for the controls, and the values for the derivatives of e above to obtain:

$$\begin{aligned} 0 &= -l + \theta c^s - (c^s + m^s) + (r - g + \delta) + (g + m^s - \delta) \\ 0 &= -l + \theta c^s + r - c^s \\ \Rightarrow l &= \theta c^s \end{aligned}$$

Consequently, the collateral constraint binds, and the Lagrange multiplier on the collateral constraint equals the debt tax shield.

6.4. Deriving Equity HJB By Assuming Collateral Constraint Always Binds

Suppose that the collateral constraint binds, then $S_t = \alpha K_t$, and the problem can be reduced to two state variables, F_t and K_t. Given corporate taxes rates θ, the flow payoffs are:

$$\Big[\underbrace{AK_t}_{\text{cash flows}} - \underbrace{\theta(AK_t - c^u F_t)}_{\text{corporate taxes}} - \underbrace{\Phi(g_t)K_t}_{\text{investment cost}}$$

$$- \underbrace{(c^u + m^u)F_t}_{\text{unsecured debt interest \& principal}} + \underbrace{p_t B_t}_{\text{unsecured debt issuance/repurchase}} \Bigg] dt$$

$$+ \underbrace{\alpha dK_t}_{\text{secured debt net issuance}}$$

Shareholders take unsecured debt price as given and solve the optimal control problem to maximize equity value, by choosing investment rate, unsecured debt issuance, and default timing.

$$E(K,F) = \max_{\tau,g,B} \mathbb{E}\Bigg[\int_0^\tau \exp(-rt)[AK_t - \theta(AK_t - c^u F_t) - \Phi(g_t)K_t - (c^u + m^u)F_t + p_t B_t]dt$$

$$+\alpha \int_0^\tau \exp(-rt) dK_t \Bigg| K_0 = K, F_0 = F \Bigg]$$

$$s.t.$$

$$\frac{dK_t}{K_t} = (g_t - \delta)dt + \sigma dZ_t$$

$$dF_t = -m^u F_t dt + B_t dt$$

where τ is the optimal stopping time for equity to default on its secured and unsecured debt obligations and cease operations.

The last term in the objective function captures the net cumulative proceeds from secured debt operations. To simplify, substitute in the expression for dK_t and then appeal to the linearity of the expectations operator and apply the stochastic version of Fubini's Theorem[21] to obtain:

$$E(K,F) = \max_{\tau,g,B} \mathbb{E}\Bigg[\int_0^\tau \exp(-rt)[AK_t - \theta(AK_t - c^u F_t) - \Phi(g_t)K_t - (c^u + m^u)F_t + p_t B_t$$

$$+\alpha(g_t - \delta)K_t]dt \Bigg| K_0 = K, F_0 = F \Bigg]$$

$$s.t.$$

$$\frac{dK_t}{K_t} = (g_t - \delta)dt + \sigma dZ_t$$

$$dF_t = -m^u F_t dt + B_t dt$$

[21] By the optional stopping theorem, τ is almost surely bounded above and has a finite expectation.

Hence, for secured debt operations, shareholders need only consider the endogenous drift in the evolution of capital, which is determined by the investment and depreciation rates.

Let $f_t \equiv F_t/K_t$ and $B_t \equiv B_t/K_t$ be rescaled variables per unit of capital. Notice:

$$K_t = K_0 \exp\left(\int_0^t (g_t - \delta - \frac{1}{2}\sigma^2)dt + \int_0^t \sigma dZ_t \right)$$

Substitute in K_t to the objective function and factor out $K_0 = K$ to find that equity value is homogeneous of degree 1 in capital (i.e. $E(1, F/K) = Ke(f)$). Further divide by K and employ a change of measure $dZ_t \equiv d\tilde{Z}_t + \sigma dt$ to obtain:

$$e(f) = \max_{\tau,g,b} \tilde{\mathbb{E}}\left[\int_0^\tau \exp\left(-\left(r - \int_0^t g_s ds + \delta\right) t\right) [A - \theta(A - c^u f_t) - \Phi(g_t) - (c^u + m^u) f_t + p_t b_t \right.$$
$$\left. + \alpha(g_t - \delta)] dt \middle| f_0 = f \right]$$
$$s.t.$$
$$df_t = (b_t - (g_t - \delta + m^u) f_t) dt - f_t \sigma d\tilde{Z}_t$$

Hence, the recursive HJB formulation of the non-recursive problem in the continuation region where the firm is in operation is given by:

$$0 = \max_{b,g} \left\{ -(r - g + \delta)e + A - \theta(A - c^u f) - \Phi(g) - (c^u + m^u) f + pb + \alpha(g - \delta) \right.$$
$$\left. + [b - (g - \delta + m^u) f] e_f + \frac{1}{2}\sigma^2 f^2 e_{ff} \right\}$$

6.5. Binding Collateral Constraints in Rampini and Viswanathan (2010)

In this section, I show that collateral constraints bind in the setup of Rampini and Viswanathan (2010) with a debt tax shield on interest expenses and unconstrained dividends (where negative dividends represent inflows to the firm from equity investors). With unconstrained dividends, the motive for the firm to conserve debt capacity is diminished. Issuing debt allows the firm to enjoy the benefit of the debt tax shield and since debt is risk-free, the firm issues up to the collateral constraint. This holds even with state-contingent debt and collateral constraints.

The firm maximizes the sum of discounted dividends over three periods ($t = 0, 1, 2$) by choosing the value of state-contingent dividends ($d_t(s)$) in each period, as well as, capital ($k_t(s)$) and the issuance of debt ($b_t(s)$) which matures in one period, where states are denoted by $s \in \mathcal{S}$. All agents are risk-neutral with discount factor $\beta \in (0, 1)$, and lenders price state-contingent debt competitively, such that, the interest rate on debt is equal to the gross-risk free rate, $R \equiv \beta^{-1} > 1$. State-contingent debt is issued against state-contingent collateral constraints: $q_t(s)\theta k_{t-1}(s) \geq Rb_t(s)$, where $q_t(s)$ is the state-contingent capital price and θ is the fraction of capital value that can be pledged as collateral.

Capital has a nonnegativity constraint and is used for production in the period ahead, such that, output is given by $A_t(s)f(k_{t-1}(s))$, where f denotes the firms production technology and is scaled by a factor A. The firm pays taxes, $\tau > 0$, on output less interest expenses, where $r \equiv R - 1$ denotes the net interest rate. Finally, the firm is subject to wealth constraints in each period and state, whereby the dividends, capital expenditures, and debt servicing costs cannot exceed output and any new borrowing.

The firm's problem is given by

$$\max_{b,d,k} \left(d_0 + \beta\mathbb{E}[d_1] + \beta^2\mathbb{E}[d_2] \right)$$

subject to constraints in period 0, 1, and 2.

Period 0 constraints:

$$w_0 \geq d_0 + q_0 k_0 - b_1(s)$$

$$k_0 \geq 0$$

where w_0 is the given level of initial wealth.

Period 1 constraints:

$$q_1(s)\theta k_0 \geq Rb_1(s)$$

$$A_1(s)f(k_0) + q_1(s)k_0 - \tau(A_1(s)f(k_0) - rb_1(s)) \geq d_1(s) + q_1(s)k_1(s) - b_2(s) + Rb_1(s)$$

$$k_1(s) \geq 0$$

Period 2 constraints:

$$q_2(s)\theta k_1(s) \geq Rb_2(s)$$

$$A_2(s)f(k_1(s)) + q_2(s)k_1(s) - \tau(A_2(s)f(k_1(s)) - rb_2(s)) \geq d_2(s) + Rb_2(s)$$

$$k_2(s) = 0$$

Denote by $\pi(s)$ the probability of realizing state s. Let the multipliers on the wealth constraints be μ_0, $\pi(s)\mu_1(s)$, and $\pi(s)\mu_2(s)$. Similarly, let the multipliers on the collateral constraints be $\pi(s)\lambda_1(s)$ and $\pi(s)\lambda_2(s)$. Finally, let the multipliers on the nonnegativity constraints for capital be ν_0^k and $\pi(s)\nu_1^k(s)$. Then, the Lagrangian for the firm's problem is given by:

$$\begin{gathered}
\mathcal{L} = d_0 + \beta \sum \pi(s) d_1(s) + \beta^2 \sum \pi(s) d_2(s) \\
-\mu_0(d_0 + q_0 k_0 - \sum \pi(s) b_1(s) - w_0) \\
-\sum \pi(s)\mu_1(s)(d_1(s) + q_1(s)k_1(s) - b_2(s) + Rb_1(s) - A_1(s)f(k_0) - q_1(s)k_0 + \tau(A_1(s)f(k_0) - rb_1(s))) \\
-\sum \pi(s)\mu_2(s)(d_2(s) + Rb_2(s) - A_2(s)f(k_1(s)) - q_2(s)k_1(s) + \tau(A_2(s)f(k_1(s)) - rb_2(s))) \\
-\sum \pi(s)\lambda_1(s)(Rb_1(s) - q_1(s)\theta k_0) \\
-\sum \pi(s)\lambda_2(s)(Rb_2(s) - q_2(s)\theta k_1(s)) \\
+\nu_0^k k_0 + \sum \pi(s)\nu_1^k(s) k_1(s)
\end{gathered}$$

Computing the derivatives of the Lagrangian with respect to dividends yields:

$$\begin{gathered}
\mu_0 = 1 \\
\mu_1(s) = \beta, \forall s \in \mathcal{S} \\
\mu_2(s) = \beta^2, \forall s \in \mathcal{S}
\end{gathered}$$

And computing the derivatives of the Lagrangian with respect to debt yields:

$$\begin{gathered}
\mu_0 = (R - \tau r)\mu_1(s) + R\lambda_1(s), \forall s \in \mathcal{S} \\
\mu_1(s) = (R - \tau r)\mu_2(s) + R\lambda_2(s), \forall s \in \mathcal{S}
\end{gathered}$$

Solving for $\lambda_1(s)$ and $\lambda_2(s)$ yields:

$$\begin{gathered}
\lambda_1(s) = \tau r \beta^2 > 0 \\
\lambda_2(s) = \tau r \beta^3 > 0
\end{gathered}$$

Hence, the Lagrange multipliers on the collateral constraints are proportional to the interest tax shield and are strictly positive. This implies that the collateral constraints bind for each time period and state.

6.6. Crisis Collateral Constraint in Model with Short-Term Secured Debt

This section provides the proof for Proposition 2.

Proof. The crisis HJB characterizing the joint value of equity and short-term debt in the continuation region is given by:

$$
\begin{aligned}
0 = \max_{b^u, b^s, g} \Big\{ & -(r - g + \delta + \lambda) j + \eta A - \theta(\eta A - c^u f - c^s s) - \Phi(g) \\
& -(c^u + m^u) f + p b^u - c^s s + p_s^* b^s - (1 - p_s^*) s \\
& + [b^u - (g + m^u - \delta) f] j_f + \frac{1}{2}\sigma^2 f^2 j_{ff} \\
& + [b^s - (g - \delta) s] j_s + \frac{1}{2}\sigma^2 s^2 j_{ss} \\
& + \lambda \bar{j} - l_s (s - \alpha) \Big\}
\end{aligned}
$$

where p_s^* is the exogenous price of short-term secured debt and l_s is the Lagrange multiplier on the constraint $s \leq \alpha$.

FOC with respect to b^s:

$$
\begin{aligned}
0 &= p_s^* + j_s \\
\Rightarrow j_s &= -p_s^* \\
\Rightarrow j_{ss} &= 0 \\
\Rightarrow j_{sf} &= 0 \\
\Rightarrow j_{sss} &= 0
\end{aligned}
$$

To obtain p_s, compute the FOC with respect to b^u to obtain $p = -j_f$. Then, differentiate with respect to s and use $j_{sf} = 0$ to conclude $p_s = 0$.

Take FOC with respect to s, using envelope condition for controls, recalling $\bar{j}_s = -1$ and $c^s = r$:

$$
\begin{aligned}
0 &= -(r - g + \delta + \lambda) j_s + \theta c^s - c^s - (1 - p_s^*) - (g - \delta) j_s + \lambda \bar{j}_s - l_s \\
0 &= (r - g + \delta + \lambda) p_s^* + \theta c^s - c^s - (1 - p_s^*) + (g - \delta) p_s^* - \lambda - l_s \\
\Rightarrow l_s &= r p_s^* + \lambda p_s^* + \theta r - r - 1 + p_s^* - \lambda \\
&= p_s^*(1 + r + \lambda) + \theta r - (1 + r + \lambda)
\end{aligned}
$$

The collateral constraint binds when $l_s > 0$, this occurs when:

$$p_s^*(1+r+\lambda)+\theta r-(1+r+\lambda)>0$$
$$p_s^*+\frac{\theta r}{1+r+\lambda}>1$$
$$\frac{\theta r}{1+r+\lambda}>1-p_s^*$$

Assuming this holds, the collateral constraint binds and $s=\alpha$. Plugging this back in and using the equilibrium conditions for the derivatives of j gives:

$$\begin{aligned}0=\max_{b^u,g}\Big\{&-(r-g+\delta+\lambda)j+\eta A-\theta(\eta A-c^u f-c^s\alpha)-\Phi(g)\\&-(c^u+m^u)f+pb^u-c^s\alpha+\alpha(p_s^*-1)+p_s^*\alpha(g-\delta)+\lambda\bar{j}\\&+[b^u-(g+m^u-\delta)f]j_f+\frac{1}{2}\sigma^2 f^2 j_{ff}\Big\}\end{aligned}$$

In case the collateral condition does not bind, $l_s=0$, then the firm does not issue short-term debt and $s=0$. The crisis HJB then becomes:

$$\begin{aligned}0=\max_{b^u,g}\Big\{&-(r-g+\delta+\lambda)j+\eta A-\theta(\eta A-c^u f)-\Phi(g)\\&-(c^u+m^u)f+pb^u+\lambda\bar{j}\\&+[b^u-(g+m^u-\delta)f]j_f+\frac{1}{2}\sigma^2 f^2 j_{ff}\Big\}\end{aligned}$$

□

6.7. Secured Debt Intervention Strictly Increases Value Function

This section provides the proof for Proposition 3.

Proof. Let $b^u(f,p_s^*)$ and $g(f,p_s^*)$ denote the optimal policies for unsecured debt issuance and investment when debt-to-capital is f and the exogenous price of secured debt is p_s^*. Let $j(f,p_s^*;p_s^{*\prime})$ be the joint value of equity and short-term debt when optimal policies are given by the arguments (f,p_s^*), but the proceeds of secured debt issuance are determined by $p_s^{*\prime}$. Moreover, the derivatives of j with respect to f are evaluated for the function j when debt-to-capital is f and secured debt price is p_s^*.

Secured debt intervention is identified as the case when $p_s^{*\prime}$ is greater than p_s^*. Assuming that the collateral constraint binds at $p_s^{*\prime}$, then we can simplify the proof that the value function increases in intervention by assuming the collateral constraints also bind at p_s^*. If the collateral constraints do not bind at $p_s^{*\prime}$, then by Equation 6, they also would not bind at p_s^*, and it is no longer the case that secured debt intervention increases the value function.

The joint equity and short-term debt HJB during a crisis, given by Equation 7, can be restated as:

$$
\begin{aligned}
j(f, p_s^*; p_s^*) = \max_{b^u, g} \Bigg\{ & \frac{1}{r - g + \delta + \lambda} \Bigg[\eta A - \theta(\eta A - c^u f - c^s \alpha) - \Phi(g) \\
& - (c^u + m^u) f + p b^u - c^s \alpha + \alpha(p_s^* - 1) + p_s^* \alpha(g - \delta) + \lambda \bar{j} \\
& + [b^u - (g + m^u - \delta) f] j_f + \frac{1}{2}\sigma^2 f^2 j_{ff} \Bigg] \Bigg\} \\
= & \frac{1}{r - g(f, p_s^*) + \delta + \lambda} \Bigg[\eta A - \theta(\eta A - c^u f - c^s \alpha) - \Phi(g(f, p_s^*)) \\
& - (c^u + m^u) f + p b^u(f, p_s^*) - c^s \alpha + \alpha(p_s^* - 1) + p_s^* \alpha(g(f, p_s^*) - \delta) + \lambda \bar{j} \\
& + [b^u(f, p_s^*) - (g(f, p_s^*) + m^u - \delta) f] j_f + \frac{1}{2}\sigma^2 f^2 j_{ff} \Bigg]
\end{aligned}
$$

By construction, $j(f, p_s^*; p_s^{*\prime}) \leq j(f, p_s^{*\prime}; p_s^{*\prime})$. Compare $j(f, p_s^*; p_s^{*\prime})$ and $j(f, p_s^*; p_s^*)$, where $p_s^{*\prime} - p_s^* > 0$:

$$
\begin{aligned}
j(f, p_s^*; p_s^{*\prime}) - j(f, p_s^*; p_s^*) &\propto \alpha(p_s^{*\prime} - 1) - \alpha(p_s^* - 1) + (p_s^{*\prime} - p_s^*)\alpha(g(f, p_s^*) - \delta) \\
&= \alpha[(p_s^{*\prime} - 1) - (p_s^* - 1)] + (p_s^{*\prime} - p_s^*)\alpha(g(f, p_s^*) - \delta) \\
&= \alpha(p_s^{*\prime} - p_s^*) + (p_s^{*\prime} - p_s^*)\alpha(g(f, p_s^*) - \delta) \\
&= (\alpha(p_s^{*\prime} - p_s^*))(1 + g(f, p_s^*) - \delta)
\end{aligned}
$$

Hence, the difference is the product of two terms. Since $\alpha \in (0, 1)$ and $p_s^{*\prime} - p_s^* > 0$, the first term is positive: $\alpha(p_s^{*\prime} - p_s^*) > 0$.

To determine the sign of the second term, first differentiate the investment policy function with respect to f yields $g_f(f, p_s^*) = \frac{1}{\gamma}(j_f - f j_{ff} - j_f) = \frac{1}{\gamma}(-f j_{ff}$. By Proposition A1, j is strictly convex. Thus, $j_{ff} > 0$ and investment decreases as f increases, attaining a minimum at the default threshold $\bar{f}$. In the default region, $j(\bar{f}) = 0$ and $j_f(\bar{f}) = 0$. Thus, the minimum value of investment is strictly positive: $g^{\min} = \frac{1}{\gamma} p_s^* \alpha > 0$. Furthermore, $\delta \in (0, 1)$, $1 - \delta > 0$. Therefore, $1 + g(f, p_s^*) - \delta > 0$.

Consequently, $j(f, p_s^*; p_s^{*\prime}) > j(f, p_s^*; p_s^*)$. All together:

$$j(f, p_s^{*\prime}; p_s^{*\prime}) \geq j(f, p_s^*; p_s^{*\prime}) > j(f, p_s^*; p_s^*)$$

Hence, secured debt intervention, such that $p_s^{*\prime} - p_s^* > 0$, strictly increases the joint value of equity and short-term debt. Since equity value is given by $e = j - \alpha$, it is also the case that:

$$e(f, p_s^{*\prime}; p_s^{*\prime}) \geq e(f, p_s^*; p_s^{*\prime}) > e(f, p_s^*; p_s^*)$$

Thus, secured debt intervention strictly increases the value of equity.

□

6.8. Secured Debt Intervention Boosts Investment Via Direct and Indirect Channels

This section provides the proof for Proposition 4.

Proof. Following the notation in Appendix 6.7, the investment policy $g(f, p_s^{*\prime})$ and $g(f, p_s^*)$ are given by:

$$\begin{aligned} g(f, p_s^{*\prime}) &= \frac{1}{\gamma}(j(f, p_s^{*\prime}; p_s^{*\prime}) - f j_f(f, p_s^{*\prime}; p_s^{*\prime}) + p_s^{*\prime}\alpha) \\ &= \frac{1}{\gamma}(j(f, p_s^{*\prime}; p_s^{*\prime}) + f p(f, p_s^{*\prime}; p_s^{*\prime}) + p_s^{*\prime}\alpha) \\ g(f, p_s^*) &= \frac{1}{\gamma}(j(f, p_s^*; p_s^*) - f j_f(f, p_s^*; p_s^*) + p_s^*\alpha) \\ &= \frac{1}{\gamma}(j(f, p_s^*; p_s^*) + f p(f, p_s^*; p_s^*) + p_s^*\alpha) \end{aligned}$$

in the case where $p_s^{*\prime} > p_s^*$, and the collateral constraints bind for both $p_s^{*\prime}$ and p_s^*. Note that the optimality condition $p = -j_f$, where p is the price of unsecured debt, is used in each case to restate the expressions.

The difference between the two policies can be expressed as:

$$g(f, p_s^{*\prime}) - g(f, p_s^*) = \frac{1}{\gamma}\begin{bmatrix} (j(f, p_s^{*\prime}; p_s^{*\prime}) + f p(f, p_s^{*\prime}; p_s^{*\prime}) + p_s^{*\prime}\alpha) \\ -(j(f, p_s^*; p_s^*) + f p(f, p_s^*; p_s^*) + p_s^*\alpha) \end{bmatrix}$$

$$= \frac{1}{\gamma} \left[\begin{array}{c} \underbrace{j(f, p_s^{*\prime}; p_s^{*\prime}) - j(f, p_s^*; p_s^*) + f(p(f, p_s^{*\prime}; p_s^{*\prime}) - p(f, p_s^*; p_s^*))}_{\text{indirect channel}} \\ + \underbrace{(p_s^{*\prime} - p_s^*)\alpha}_{\text{direct channel}} \end{array} \right]$$

The change in investment can be decomposed into a direct channel and an indirect channel. Since $\gamma > 0$ and $p_s^{*\prime} > p_s^*$, there is a direct increase to investment arising from greater proceeds from secured debt issuance.

Indirectly, an increase in the joint value for equity and short-term debt and an increase in the price of unsecured debt can lead to higher investment. To see this, first note that by Proposition 3, as shown in Appendix 6.7, $j(f, p_s^{*\prime}; p_s^{*\prime}) > j(f, p_s^*; p_s^*)$. Second, recall that the price of unsecured debt, p, as defined in Section 2.4, is increasing in the default time τ. Since $j(f, p_s^{*\prime}; p_s^{*\prime}) > j(f, p_s^*; p_s^*)$, $\tau(f, p_s^{*\prime}; p_s^{*\prime}) \geq \tau(f, p_s^*; p_s^*)$ and so, $p(f, p_s^{*\prime}; p_s^{*\prime}) - p(f, p_s^*; p_s^*) \geq 0$. Equivalently, recalling the definition for Tobin's q, we can conclude that $q(f, p_s^{*\prime}; p_s^{*\prime}) > q(f, p_s^*; p_s^*)$; that is, Tobin's q is higher under secured debt intervention.

In sum, $g(f, p_s^{*\prime}) - g(f, p_s^*) > 0$ and investment is strictly increasing in secured debt intervention. This increase can be decomposed into direct and indirect channels. □

6.9. Segmented Markets in Model with Short-Term Secured Debt

This section provides the proof for Proposition 5.

Proof. First, we verify that changing unsecured debt issuance does not change the joint equity and short-term debt value function. This can be seen from differentiating Equation (7) with respect to b^u and obtaining the optimality condition $p = -j_f$. Plugging this into (7) shows that the value function is invariant to the choice of b^u, although the equilibrium values of b^u are different in the case without and with unsecured debt intervention.

Given the above, differentiate the no-trade crisis state HJB equation for equity and short-term debt, as characterized by Equation (7) with $b^u = 0$, with respect to f and where the discount rate is given by $r^{(e)}$, using the envelope condition for investment:

$$\begin{aligned} (r^{(e)} - g^* + \delta + \lambda) j_f &= \theta c^u - (c^u + m^u) - (g^* + m^u - \delta) f j_{ff} - (g^* + m^u - \delta) e_f \\ &\quad + \sigma^2 f j_{ff} + \frac{1}{2}\sigma^2 f^2 j_{fff} + \lambda \bar{e}_f \\ \Rightarrow (r^{(e)} + m^u + \lambda) j_f &= \theta c^u - (c^u + m^u) - (g^* + m^u - \delta - \sigma^2) f j_{ff} + \frac{1}{2}\sigma^2 f^2 j_{fff} + \lambda \bar{j}_f \end{aligned}$$

Substitute in the first order condition for b^u, $p = -j_f$ and $\bar{p} = -\bar{j}_f$, into the crisis HJB equation for unsecured debt price, as in Equation (8), but priced with discount rate r^d:

$$-(r^{(d)} + m^u + \lambda) j_f = c^u + m^u - [b^u - (g^* + m^u - \delta - \sigma^2) f] j_{ff} - \frac{1}{2}\sigma^2 f^2 j_{fff} - \lambda \bar{j}_f$$

Add these two expression together:

$$\begin{aligned}(r^{(e)} - r^{(d)}) j_f &= \theta c^u - b^u j_{ff} \\ \Rightarrow b^u &= \frac{\theta c^u}{j_{ff}} - \frac{(r^{(e)} - r^{(d)}) j_f}{j_{ff}}\end{aligned}$$

□

6.10. Unsecured Debt Intervention and Joint Equity and Short-Term Debt Value Function

This section provides the proof for Proposition 6.

Proof. Appendix 6.9 verifies that the joint equity and short-term debt value functions are invariant to unsecured debt issuance policy and derives the general expression for this policy in segmented markets.

Without unsecured debt intervention, $r^{(e)} = r^{(d)}$ and:

$$b^{\text{no int}} = \frac{\theta c^u}{j_{ff}}$$

In the case of unsecured debt intervention, $r^{(e)} > r^{(d)}$ and

$$\begin{aligned}b^{\text{int}} &= \frac{\theta c^u}{j_{ff}} - \frac{(r^{(e)} - r^{(d)}) j_f}{j_{ff}} \\ &= \frac{\theta c^u}{j_{ff}} + \frac{(r^{(e)} - r^{(d)}) p}{j_{ff}}\end{aligned}$$

The difference between the issuance policies is given by:

$$b^{\text{int}} - b^{\text{no int}} = \frac{(r^{(e)} - r^{(d)}) p}{j_{ff}} > 0$$

which is strictly positive in the continuation region because debt price $p > 0$ and j is convex.

Given that j is invariant to issuance policy, the optimal investment policy is invariant to issuance and is denoted as $g^*(f)$. Let $j(f; b^{\text{int}})$ denote the value function given issuance policy under unsecured debt intervention; similarly, denote $j(f; b^{\text{no int}})$. The difference between these two value function is equal to zero and given by:

$$j(f; b^{\text{int}}) - j(f; b^{\text{no int}}) \propto pb^{\text{int}} - pb^{\text{no int}} + \mathcal{A}(b^{\text{int}})j(f; b^{\text{int}}) - \mathcal{A}(b^{\text{no int}})j(f; b^{\text{no int}}) = 0$$

where $\mathcal{A}$ is the infinitesimal generator of j, such that $\mathcal{A}(b^u)j = [b^u - (g + m^u - \delta)f]j_f + \frac{1}{2}\sigma^2 f^2 j_{ff}$ is the continuation value of j. Rearranging, we have:

$$p(b^{\text{int}} - b^{\text{no int}}) = \mathcal{A}(b^{\text{no int}})j(f; b^{\text{no int}}) - \mathcal{A}(b^{\text{int}})j(f; b^{\text{int}})$$

Since $b^{\text{int}} - b^{\text{no int}} > 0$ and $p > 0$ in the continuation region, the continuation value under $b^{\text{no int}}$ is strictly greater than the continuation value under b^{int}: $\mathcal{A}(b^{\text{no int}})j(f; b^{\text{no int}}) > \mathcal{A}(b^{\text{int}})j(f; b^{\text{int}})$.

Consequently, unsecured debt intervention accelerates payouts immediately by stimulating greater issuance at the cost of a lower continuation value. This also suggests implications for the evolution of the distribution of firms over the state space, that is, the distribution of firms' unsecured debt-to-capital. Heuristically, a lower continuation value implies higher levels of leverage, which in turn implies higher default rates. Numerical results confirm this conjecture.

□

6.11. Dividend Restriction in Model with Short-Term Secured Debt

This section provides the proof for Proposition 7.

6.11.1. Complementary Slackness Condition

- The complementary slackness condition corresponding to the constraint given by Equation (9) and Lagrange multiplier l is:

$$l\pi(b^u, g) = 0$$
$$l \geq 0, \pi(b^u, g) \leq 0$$

Then,

$$\pi < 0 \Rightarrow l = 0$$
$$\pi = 0 \Rightarrow l > 0$$

Note:

$$\begin{aligned}
\pi_{b^u} &= p \\
\pi_g &= -\Phi'(g) + p_s^* \alpha \\
\pi_f &= \theta c^u - (c^u + m^u) + p_f b^u
\end{aligned}$$

6.11.2. FOCs for Joint Equity and Short-Term Debt HJB

- Taking the Lagrange multiplier l as given, the control problem for shareholders becomes:

$$0 = \max_{b^u, g} \left\{ -(r - g + \delta + \lambda) j + (1 - l)\pi(b^u, g) + \lambda \bar{j} + [b^u - (g - \delta + m^u) f] j_f + \frac{1}{2}\sigma^2 f^2 j_{ff} \right\}$$

- FOC w/r/t g:

$$\begin{aligned}
0 &= j - f j_f + (1 - l)\pi_g \\
&= j - f j_f - (1 - l)\Phi'(g) + (1 - l) p_s^* \alpha \\
\Rightarrow \Phi'(g) &= \frac{j - f j_f}{1 - l} + p_s^* \alpha \\
g &= \frac{1}{\gamma}\left(\frac{j - f j_f}{1 - l} + p_s^* \alpha \right)
\end{aligned}$$

- FOC w/r/t b^u:

$$\begin{aligned}
0 &= j_f + (1 - l)\pi_{b^u} \\
-(1 - l)\pi_{b^u} &= j_f \\
\Rightarrow p &= -\frac{j_f}{1 - l} \\
\Rightarrow l &= 1 + \frac{j_f}{p}
\end{aligned}$$

Note:

$$p_f = -\frac{j_{ff}}{1-l}$$
$$p_{ff} = -\frac{j_{fff}}{1-l}$$

6.11.3. Optimal Issuance

- The crisis HJB for debt is:

$$(r + m^u + \lambda)\, p = c^u + m^u + \lambda \bar{p} + [b - (g + m^u - \delta - \sigma^2) f]\, p_f + \frac{1}{2}\sigma^2 f^2 p_{ff}$$

- Substitute in the expression for p, p_f, p_{ff} and $\bar{p}$:

$$(r + m^u + \lambda)\left(-\frac{j_f}{1-l}\right) = c^u + m^u - \lambda \bar{j}_f + [b^u - (g + m^u - \delta - \sigma^2) f]\left(-\frac{j_{ff}}{1-l}\right) + \frac{1}{2}\sigma^2 f^2 \left(-\frac{j_{fff}}{1-l}\right)$$
$$\Rightarrow -(r + m^u + \lambda)\, j_f = (1-l)(c^u + m^u) - (1-l)\lambda \bar{j}_f - b^u j_{ff} + (g + m^u - \delta - \sigma^2) f j_{ff} - \frac{1}{2}\sigma^2 f^2 j_{fff}$$

- Take derivative of crisis equity HJB w/r/t f:

$$\begin{aligned}
0 = &- (r - g + \delta + \lambda)\, j_f + (1-l)\pi_f + \lambda \bar{j}_f \\
&+ b^u j_{ff} - (g - \delta + m^u) j_f - (g - \delta + m^u) f j_{ff} + \sigma^2 f j_{ff} + \frac{1}{2}\sigma^2 f^2 j_{fff} \\
\Rightarrow (r + m^u + \lambda)\, j_f &= (1-l)\pi_f + \lambda \bar{j}_f + [b^u - (g - \delta + m^u - \sigma^2) f]\, j_{ff} + \frac{1}{2}\sigma^2 f^2 j_{fff} \\
&= (1-l)(\theta c^u - (c^u + m^u) + p_f b^u) + \lambda \bar{j}_f \\
&+ [b^u - (g - \delta + m^u - \sigma^2) f]\, j_{ff} + \frac{1}{2}\sigma^2 f^2 j_{fff} \\
&= (1-l)\theta c^u - (1-l)(c^u + m^u) - j_{ff} b^u + \lambda \bar{j}_f \\
&+ [b^u - (g - \delta + m^u - \sigma^2) f]\, j_{ff} + \frac{1}{2}\sigma^2 f^2 j_{fff} \\
&= (1-l)\theta c^u - (1-l)(c^u + m^u) + \lambda \bar{j}_f - (g - \delta + m^u - \sigma^2) f j_{ff} + \frac{1}{2}\sigma^2 f^2 j_{fff}
\end{aligned}$$

- Add these two equations together:

$$0 = (1-l)\theta c^u + l \lambda \bar{j}_f - b^u j_{ff}$$

$$
\begin{aligned}
\Rightarrow b^u &= (1-l)\frac{\theta c^u}{j_{ff}} + l\frac{\lambda \bar{j}_f}{j_{ff}} \\
&= (1-l)\frac{\theta c^u}{j_{ff}} - l\frac{\lambda \bar{p}}{j_{ff}}
\end{aligned}
$$

6.11.4. Solution with Binding Constraint

- When constraint binds:

$$
\begin{aligned}
\pi(b^u, g) &= -\frac{1}{2}\gamma g^2 + p_s^* \alpha g + \eta A - \theta(\eta A - c^u f - c^s \alpha) - (c^u + m^u) f + p b^u - c^s \alpha + (p_s^* - 1)\alpha - p_s^* \alpha \delta = 0 \\
\zeta(b^u) &\equiv \eta A - \theta(\eta A - c^u f - c^s \alpha) - (c^u + m^u) f + p b^u - c^s \alpha + (p_s^* - 1)\alpha - p_s^* \alpha \delta \\
\Rightarrow g &= \frac{-p_s^* \alpha - \sqrt{(p_s^* \alpha)^2 + 2\gamma \zeta(b)}}{-\gamma} \\
&= \frac{p_s^* \alpha + \sqrt{(p_s^* \alpha)^2 + 2\gamma \zeta(b)}}{\gamma}
\end{aligned}
$$

- Investment increases in unsecured debt issuance:

$$
\begin{aligned}
\frac{\partial g}{\partial b^u} &= \frac{1}{\gamma}\frac{1}{2}((p_s^* \alpha)^2 + 2\gamma \zeta(b^u))^{-1/2}(2\gamma)\zeta'(b^u) \\
&= \frac{p}{\sqrt{(p_s^* \alpha)^2 + 2\gamma \zeta(b^u)}}
\end{aligned}
$$

- Ensuring that investment is non-negative requires $\zeta(b^u) \geq 0$:

$$
\begin{aligned}
0 &\leq \zeta(b^u) \\
&\leq \eta A - \theta(\eta A - c^u f - c^s \alpha) - (c^u + m^u) f + p b^u - c^s \alpha + (p_s^* - 1)\alpha - p_s^* \alpha \delta \\
\Rightarrow b^u &\geq -\frac{1}{p}(\eta A - \theta(\eta A - c^u f - c^s \alpha) - (c^u + m^u) f - c^s \alpha + (p_s^* - 1)\alpha - p_s^* \alpha \delta)
\end{aligned}
$$

- I also consider a restriction on unsecured debt repurchases while the dividend restriction is in place (i.e. $b^u \geq 0$). This ensures strictly positive investment.

6.12. Dividend Restriction and Unsecured Debt Repurchase Restriction in Model with Short-Term Secured Debt

Let l_1 be the Lagrange multiplier on the dividend constraint Equation (9) and l_2 be the Lagrange multiplier on the unsecured debt issuance constraint: $b^u \geq 0$.

Then, the complementary complementary slackness condition for the dividend constraint is:

$$l_1 \pi(b^u, g) = 0$$
$$l_1 \geq 0, \pi(b^u, g) \leq 0$$

and the complementary slackness condition for the unsecured debt issuance constraint is:

$$-l_2 b^u = 0$$
$$l_2 \geq 0, -b^u \leq 0$$

Using the Lagrange multipliers, l_1 and l_2, the control problem in the continuation region can be express as:

$$0 = \max_{b^u, g} \left\{ -(r - g + \delta + \lambda) j + (1 - l_1)\pi(b^u, g) + l_2 b^u + \lambda \bar{j} + [b^u - (g - \delta + m^u) f] j_f + \frac{1}{2}\sigma^2 f^2 j_{ff} \right\}$$

The FOC with respect to optimal investment, g, is:

$$\begin{aligned} 0 &= j - f j_f + (1 - l_1)\pi_g \\ &= j - f j_f - (1 - l_1)\Phi'(g) + (1 - l_1) p_s^* \alpha \\ \Rightarrow \Phi'(g) &= \frac{j - f j_f}{1 - l_1} + p_s^* \alpha \\ g &= \frac{1}{\gamma} \left(\frac{j - f j_f}{1 - l_1} + p_s^* \alpha \right) \end{aligned}$$

An expression for l_1 can be obtained in terms of g:

$$\begin{aligned} g\gamma &= \frac{j - f j_f}{1 - l_1} + p_s^* \alpha \\ g\gamma(1 - l_1) &= j - f j_f + (1 - l_1) p_s^* \alpha \\ l_1(p_s^* \alpha - g\gamma) &= j - f j_f + p_s^* \alpha - g\gamma \\ l_1 &= 1 + \frac{j - f j_f}{p_s^* \alpha - g\gamma} \end{aligned}$$

The FOC with respect to optimal unsecured debt issuance, b^u, is:

$$
\begin{aligned}
0 &= j_f + (1 - l_1)\pi_{b^u} + l_2 \\
-(1 - l)\pi_{b^u} &= j_f + l_2 \\
\Rightarrow\ p &= -\frac{j_f + l_2}{1 - l_1} \\
\Rightarrow l_1 &= 1 + \frac{j_f + l_2}{p}
\end{aligned}
$$

If the dividend constraint binds, $l_1 > 0$, then:

$$
\begin{aligned}
0 &< p + j_f + l_2 \\
-p - j_f &< l_2
\end{aligned}
$$

If the dividend constraint doesn't bind, $l_1 = 0$, then:

$$
0 = p + j_f + l_2
$$

Note:

$$
\begin{aligned}
p_f &= -\frac{j_{ff}}{1 - l_1} \\
p_{ff} &= -\frac{j_{fff}}{1 - l_1}
\end{aligned}
$$

The crisis HJB for debt is:

$$
(r + m^u + \lambda)\, p = c^u + m^u + \lambda \bar{p} + [b^u - (g + m^u - \delta - \sigma^2) f]\, p_f + \frac{1}{2}\sigma^2 f^2\, p_{ff}
$$

Substitute in the expression for p, p_f, p_{ff} and $\bar{p}$:

$$
\begin{aligned}
(r + m^u + \lambda)\left(-\frac{j_f + l_2}{1 - l_1}\right) &= c^u + m^u - \lambda \bar{j}_f + [b^u - (g + m^u - \delta - \sigma^2) f]\left(-\frac{j_{ff}}{1 - l_1}\right) + \frac{1}{2}\sigma^2 f^2 \left(-\frac{j_{fff}}{1 - l_1}\right) \\
\Rightarrow -(r + m^u + \lambda)(j_f + l_2) &= (1 - l_1)(c^u + m^u) - (1 - l_1)\lambda \bar{j}_f \\
&\quad - b^u j_{ff} + (g + m^u - \delta - \sigma^2) f j_{ff} - \frac{1}{2}\sigma^2 f^2 j_{fff}
\end{aligned}
$$

Take derivative of crisis equity HJB w/r/t f:

$$\begin{aligned}
0 =& -(r-g+\delta+\lambda)j_f + (1-l_1)\pi_f + \lambda\bar{j}_f \\
&+ b^u j_{ff} - (g-\delta+m^u)j_f - (g-\delta+m^u)fj_{ff} + \sigma^2 fj_{ff} + \frac{1}{2}\sigma^2 f^2 j_{fff} \\
\Rightarrow (r+m^u+\lambda)j_f =& (1-l_1)\pi_f + \lambda\bar{j}_f + [b^u - (g-\delta+m^u-\sigma^2)f]j_{ff} + \frac{1}{2}\sigma^2 f^2 j_{fff} \\
=& (1-l_1)(\theta c^u - (c^u+m^u) + p_f b^u) + \lambda\bar{j}_f \\
&+ [b^u - (g-\delta+m^u-\sigma^2)f]j_{ff} + \frac{1}{2}\sigma^2 f^2 j_{fff} \\
=& (1-l_1)\theta c^u - (1-l_1)(c^u+m^u) - j_{ff}b^u + \lambda\bar{j}_f \\
&+ [b^u - (g-\delta+m^u-\sigma^2)f]j_{ff} + \frac{1}{2}\sigma^2 f^2 j_{fff} \\
=& (1-l_1)\theta c^u - (1-l_1)(c^u+m^u) + \lambda\bar{j}_f - (g-\delta+m^u-\sigma^2)fj_{ff} + \frac{1}{2}\sigma^2 f^2 j_{fff}
\end{aligned}$$

Add these two equations together:

$$\begin{aligned}
-l_2(r+m^u+\lambda) &= (1-l_1)\theta c^u + l_1\lambda\bar{j}_f - b^u j_{ff} \\
\Rightarrow b^u &= (1-l_1)\frac{\theta c^u}{j_{ff}} + l_1\frac{\lambda\bar{j}_f}{j_{ff}} + l_2\frac{r+m^u+\lambda}{j_{ff}} \\
&= (1-l_1)\frac{\theta c^u}{j_{ff}} - l_1\frac{\lambda\bar{p}}{j_{ff}} + l_2\frac{r+m^u+\lambda}{j_{ff}}
\end{aligned}$$

When the debt repurchase restriction binds, $b^u = 0$ and:

$$\begin{aligned}
l_2\frac{r+m^u+\lambda}{j_{ff}} &= l_1\frac{\lambda\bar{p}}{j_{ff}} - (1-l_1)\frac{\theta c^u}{j_{ff}} \\
l_2 &= l_1\frac{\lambda\bar{p}}{r+m^u+\lambda} - (1-l_1)\frac{\theta c^u}{r+m^u+\lambda} \geq 0
\end{aligned}$$

6.13. Numerical Solution for Model with Short-Term Debt

- Endogenous state f: $\Delta f : \{f_1, \ldots, f_I\}$, where:

$$f_1 = f_{i-1} + \Delta f = f_1 + (i-1)\Delta f$$

for $2 \leq i \leq I$, where $f_1 = 0$. Policy on boundary such that process obeys the state constraint.

6.13.1. Solution with No-Trade

Discrete Dynamics of Endogenous State Variable

- Drift for endogenous state f: $\iota(f_i) = -(g(f_i) + m^u - \delta) f_i = -(\frac{1}{\gamma}(j_i - f \partial j_i + \alpha) + m^u - \delta) f_i$.
- Forward approximation for $\partial_F j_i$:

$$\partial_F j_i \equiv \frac{j_{i+1} - j_i}{\Delta f}$$

- Backward approximation for $\partial_B j_i$:

$$\partial_B j_i \equiv \frac{j_i - j_{i-1}}{\Delta f}$$

- Second derivative (central):

$$\frac{\partial^2 j(f)}{\partial f^2} \approx \partial_{ff} j_i \equiv \frac{j_{i+1} + j_{i-1} - 2 j_i}{(\Delta f)^2}$$

- Choice of approximation depends on sign of ι_i (per upwind scheme):
  - $\iota_{i,F} = -(\frac{1}{\gamma}(j_i - f_i \partial_F j_i + \alpha) + m^u - \delta) f_i > 0 \Rightarrow g_i = \frac{1}{\gamma}(j_i - f_i \partial_F j_i + \alpha)$.
  - $\iota_{i,B} = -(\frac{1}{\gamma}(j_i - f_i \partial_B j_i + \alpha) + m^u - \delta) f_i < 0 \Rightarrow g_i = \frac{1}{\gamma}(j_i - f_i \partial_B j_i + \alpha)$.
  - $\iota_i = -(\frac{1}{\gamma}(j_i - f_i \partial_B j_i + \alpha) + m^u - \delta) f_i = 0 \Rightarrow g_i = -m^u + \delta$.

Discretized HJB Equation

- Let $\Psi_i^n = A - \theta(A - c^u f_i - c^s \alpha) - \Phi(g_i^n) - (c^u + m^u) f_i - c^s \alpha + \alpha(g_i^n - \delta)$.
- Given time step Δ, the discretized HJB equation in the continuation is given by:

$$\begin{aligned}\frac{j_i^{n+1} - j_i^n}{\Delta} + (r - g_i^n + \delta) j_i^{n+1} &= \Psi_i^n + \iota_{i,F}^n \mathbb{1}_{\iota_{i,F}^n > 0} \partial_F j_i^{n+1} + \iota_{i,B}^n \mathbb{1}_{\iota_{i,B}^n < 0} \partial_B j_i^{n+1} + \frac{1}{2} \sigma^2 f_i^2 j_{ff} \\ &= \Psi_i^n + \iota_{i,F}^n \mathbb{1}_{\iota_{i,F}^n > 0} \frac{j_{i+1}^{n+1} - j_i^{n+1}}{\Delta f} + \iota_{i,B}^n \mathbb{1}_{\iota_{i,B}^n < 0} \frac{j_i^{n+1} - j_{i-1}^{n+1}}{\Delta f} \\ &\quad + \frac{\sigma^2 f_i^2}{2} \frac{j_{i+1}^{n+1} + j_{i-1}^{n+1} - 2 j_i^{n+1}}{(\Delta f)^2}\end{aligned}$$

- Collecting terms:

$$\left(-\frac{\iota_{i,B}^{n}\mathbb{1}_{\iota_{i,B}^{n}<0}}{\Delta f}+\frac{1}{2}\frac{\sigma^{2}f_{i}^{2}}{(\Delta f)^{2}}\right)j_{i-1}^{n+1}+\left(-\frac{\iota_{i,F}^{n}\mathbb{1}_{\iota_{i,F}^{n}>0}}{\Delta f}+\frac{\iota_{i,B}^{n}\mathbb{1}_{\iota_{i,B}^{n}<0}}{\Delta f}-\frac{\sigma^{2}f_{i}^{2}}{(\Delta f)^{2}}\right)j_{i}^{n+1}$$
$$+\left(\frac{\iota_{i,F}^{n}\mathbb{1}_{\iota_{i,F}^{n}>0}}{\Delta f}+\frac{1}{2}\frac{\sigma^{2}f_{i}^{2}}{(\Delta f)^{2}}\right)j_{i+1}^{n+1}$$

- Define:

$$\begin{aligned}
\xi_i &= -\frac{\iota_{i,B}^{n}\mathbb{1}_{\iota_{i,B}^{n}<0}}{\Delta f}+\frac{1}{2}\frac{\sigma^{2}f_{i}^{2}}{(\Delta f)^{2}}\\
\beta_i &= -\frac{\iota_{i,F}^{n}\mathbb{1}_{\iota_{i,F}^{n}>0}}{\Delta f}+\frac{\iota_{i,B}^{n}\mathbb{1}_{\iota_{i,B}^{n}<0}}{\Delta f}-\frac{\sigma^{2}f_{i}^{2}}{(\Delta f)^{2}}\\
\zeta_i &= \frac{\iota_{i,F}^{n}\mathbb{1}_{\iota_{i,F}^{n}>0}}{\Delta f}+\frac{1}{2}\frac{\sigma^{2}f_{i}^{2}}{(\Delta f)^{2}}
\end{aligned}$$

- In matrix notation,

$$\frac{\mathbf{j}^{n+1}-\mathbf{j}^{n}}{\Delta}+\mathrm{diag}((r+\delta)\mathbf{1}-\mathbf{g}^{n})\mathbf{j}^{n+1}=\Psi^{n}+\mathbf{C}\mathbf{j}^{n+1}$$

where,

$$\begin{aligned}
\mathbf{j},\mathbf{1},\mathbf{g},{}^{-n},\mathbf{j}^{n+1} &\in \mathbb{R}^{I\times 1}\\
\mathbf{C} &\in \mathbb{R}^{I\times I}\\
\mathrm{diag}((r+\delta)\mathbf{1}-\mathbf{g}^{n}) &\in \mathbb{R}^{I\times I}
\end{aligned}$$

- Matrix $\mathbf{C}$ is the discrete-space approximation of the infinitesimal generator $\mathcal{C}$.
- System can also be written as:

$$\mathbf{B}^{\mathbf{n}}\mathbf{j}^{n+1}=\mathbf{b}^{n}$$

where,

$$\mathbf{B}^{n}=\mathrm{diag}\left(\left(\frac{1}{\Delta}+r+\delta\right)\mathbf{1}-\mathbf{g}^{n}\right)-\mathbf{C}^{n}$$

$$\mathbf{b}^n = \Psi^n + \frac{1}{\Delta}\mathbf{j}^n$$

- The HJBVI which takes into account the default option can be expressed in the standard form of a linear complementarity problem (LCP):

$$\begin{aligned} \mathbf{j}^{n+1\prime}(\mathbf{B}^n\mathbf{j}^{n+1} + (-\mathbf{b}^n)) = 0 \\ \mathbf{j}^{n+1} \geq 0 \\ \mathbf{B}^n\mathbf{j}^{n+1} + (-\mathbf{b}^n) \geq 0 \end{aligned}$$

Boundary Conditions

- Assuming we've solved for the equilibrium transition rate matrix, $\mathbf{C}$, take as given the default threshold, $f_D \in (f_1, f_I)$, where we assume that the state grid is sufficiently constructed so that f_D lies in the interior.

- We have the following state constraint at the lower boundary (there is no negative drift in df):

$$f \geq f_1 = 0 \Rightarrow \iota_{1,B} = 0$$

Also, $f_1 = 0 \Rightarrow \iota_{1,F} = 0$. Therefore,

$$\begin{aligned} \xi_1 &= 0 \\ \beta_1 &= 0 \\ \zeta_1 &= 0 \end{aligned}$$

- A firm entering default cannot resume operations or accumulate more debt:

$$f = f_D, \forall t \Rightarrow \begin{bmatrix} \xi_D & \beta_D & \zeta_D \end{bmatrix} = 0$$

6.13.2. Solution with Trade and No Market Segmentation

- Upwind scheme unchanged. Unsecured debt issuance policy only affects drift.

Discretized HJB Equation

- Let $\Psi_i^n = A - \theta(A - c^u f_i - c^s\alpha) - \Phi(g_i^n) - (c^u + m^u)f_i - c^s\alpha + p_i^n b_i^n + \alpha(g_i^n - \delta)$.

- Given time step Δ, the discretized HJB equation in the continuation is given by:

$$\frac{j_i^{n+1}-j_i^n}{\Delta}+(r-g_i^n+\delta)j_i^{n+1}=\Psi_i^n+b_i^n\partial_F j_i^{n+1}+\iota_{i,F}^n\mathbb{1}_{\iota_{i,F}^n>0}\partial_F j_i^{n+1}+\iota_{i,B}^n\mathbb{1}_{\iota_{i,B}^n<0}\partial_B j_i^{n+1}+\frac{1}{2}\sigma^2 f_i^2 j_{ff}$$
$$=\Psi_i^n+b_i^n\frac{j_{i+1}^{n+1}-j_i^{n+1}}{\Delta f}+\iota_{i,F}^n\mathbb{1}_{\iota_{i,F}^n>0}\frac{j_{i+1}^{n+1}-j_i^{n+1}}{\Delta f}+\iota_{i,B}^n\mathbb{1}_{\iota_{i,B}^n<0}\frac{j_i^{n+1}-j_{i-1}^{n+1}}{\Delta f}$$
$$+\frac{\sigma^2 f_i^2}{2}\frac{j_{i+1}^{n+1}+j_{i-1}^{n+1}-2j_i^{n+1}}{(\Delta f)^2}$$

- Collecting terms:

$$\left(-\frac{\iota_{i,B}^n\mathbb{1}_{\iota_{i,B}^n<0}}{\Delta f}+\frac{1}{2}\frac{\sigma^2 f_i^2}{(\Delta f)^2}\right)j_{i-1}^{n+1}+$$
$$\left(-\frac{b_i^n}{\Delta f}-\frac{\iota_{i,F}^n\mathbb{1}_{\iota_{i,F}^n>0}}{\Delta f}+\frac{\iota_{i,B}^n\mathbb{1}_{\iota_{i,B}^n<0}}{\Delta f}-\frac{\sigma^2 f_i^2}{(\Delta f)^2}\right)j_i^{n+1}+$$
$$\left(\frac{b_i^n}{\Delta f}+\frac{\iota_{i,F}^n\mathbb{1}_{\iota_{i,F}^n>0}}{\Delta f}+\frac{1}{2}\frac{\sigma^2 f_i^2}{(\Delta f)^2}\right)j_{i+1}^{n+1}$$

- Define:

$$\xi_i=-\frac{\iota_{i,B}^n\mathbb{1}_{\iota_{i,B}^n<0}}{\Delta f}+\frac{1}{2}\frac{\sigma^2 f_i^2}{(\Delta f)^2}$$
$$\beta_i=-\frac{b_i^n}{\Delta f}-\frac{\iota_{i,F}^n\mathbb{1}_{\iota_{i,F}^n>0}}{\Delta f}+\frac{\iota_{i,B}^n\mathbb{1}_{\iota_{i,B}^n<0}}{\Delta f}-\frac{\sigma^2 f_i^2}{(\Delta f)^2}$$
$$\zeta_i=\frac{b_i^n}{\Delta f}+\frac{\iota_{i,F}^n\mathbb{1}_{\iota_{i,F}^n>0}}{\Delta f}+\frac{1}{2}\frac{\sigma^2 f_i^2}{(\Delta f)^2}$$

Boundary Conditions

– We have the following state constraint at the lower boundary (there is no negative drift in df):

$$f\geq f_1=0\Rightarrow\iota_{1,B}=0$$

However, unlike before, at $f_1 = 0 \Rightarrow \iota_{1,F} = b_1^* > 0$. Therefore,

$$\begin{aligned}\xi_1 &= 0\\ \beta_1 &= -\frac{b_1^n}{\Delta f}\\ \zeta_1 &= \frac{b_1^n}{\Delta f}\end{aligned}$$

– As before, a firm entering default cannot resume operations or accumulate more debt:

$$f = f_D, \forall t \Rightarrow \begin{bmatrix}\xi_D & \beta_D & \zeta_D\end{bmatrix} = 0$$

6.13.3. Finite difference method for debt price

- With a dividend restriction, it is no longer the case that the no-trade equity value is equal to the equity value with trade.
- As a result, the debt price needs to be solved for taking equity's optimal policies as given.
- The HJB for debt in the crisis region is:

$$\begin{aligned}(r + m^u + \lambda)\, p =& c^u + m^u\\ &+ [b - (g + m^u - \delta - \sigma^2) f]\, p_f + \frac{1}{2}\sigma^2 f^2\, p_{ff} + \lambda \bar{p}\end{aligned}$$

- Then given time step Δ, the discretized HJB equation in the continuation is given by:

$$\begin{aligned}\frac{p_i^{n+1} - p_i^n}{\Delta} + (r + m^u + \lambda)\, p_i^{n+1} &= c^u + m^u + \lambda \bar{p} + [b_i^n - (g_i^n + m^u - \delta - \sigma^2) f_i]\partial\, p_i^{n+1}\\ &\quad + \frac{1}{2}\sigma^2 f_i^2 \partial^2\, p_i^{n+1}\\ &= c^u + m^u + \lambda \bar{p}\\ &\quad + [b_i^n - (g_i^n + m^u - \delta - \sigma^2) f_i]\frac{p_{i+1}^{n+1} - p_i^{n+1}}{\Delta f}\\ &\quad + \frac{\sigma^2 f_i^2}{2}\frac{p_{i+1}^{n+1} + p_{i-1}^{n+1} - 2\, p_i^{n+1}}{(\Delta f)^2}\end{aligned}$$

- Collecting terms:

$$\left(\frac{\sigma^2 f_i^2}{2(\Delta f)^2}\right) p_{i-1}^{n+1} + \left(-\frac{[b_i^n - (g_i^n + m^u - \delta - \sigma^2) f_i]}{\Delta f} - \frac{\sigma^2 f_i^2}{(\Delta f)^2}\right) p_i^{n+1}$$
$$+ \left(\frac{[b_i^n - (g_i^n + m^u - \delta - \sigma^2) f_i]}{\Delta f} + \frac{\sigma^2 f_i^2}{2(\Delta f)^2}\right) p_{i+1}^{n+1}$$

- Denote:

$$\xi_i^p = \frac{\sigma^2 f_i^2}{2(\Delta f)^2}$$
$$\beta_i^p = -\frac{[b_i^n - (g_i^n + m^u - \delta - \sigma^2) f_i]}{\Delta f} - \frac{\sigma^2 f_i^2}{(\Delta f)^2}$$
$$\zeta_i^p = \frac{[b_i^n - (g_i^n + m^u - \delta - \sigma^2) f_i]}{\Delta f} + \frac{\sigma^2 f_i^2}{2(\Delta f)^2}$$

- In matrix notation,

$$\frac{\mathbf{p}^{n+1} - \mathbf{p}^n}{\Delta} + (r + m^u + \lambda)\mathbf{p}^{n+1} = (c^u + m^u)\mathbf{1} + \lambda\bar{\mathbf{p}} + \mathbf{C}^p\mathbf{p}^{n+1}$$

where,

$$\mathbf{p}, \bar{\mathbf{p}}, \mathbf{1} \in \mathbb{R}^{I\times 1}$$
$$\mathbf{C}^p \in \mathbb{R}^{I\times I}$$

- Rewrite system:

$$\frac{\mathbf{p}^{n+1}}{\Delta} + (r + m^u + \lambda)\mathbf{p}^{n+1} - \mathbf{C}^p\mathbf{p}^{n+1} = (c^u + m^u)\mathbf{1} + \lambda\bar{\mathbf{p}} + \frac{1}{\Delta}\mathbf{p}^n$$
$$\left(\operatorname{diag}\left(\frac{1}{\Delta} + (r + m^u + \lambda)\right) - \mathbf{C}^p\right)\mathbf{p}^{n+1} = (c^u + m^u)\mathbf{1} + \lambda\bar{\mathbf{p}} + \frac{1}{\Delta}\mathbf{p}^n$$
$$\mathbf{B}^{p,n}\mathbf{p}^{n+1} = \mathbf{b}^{p,n}$$

where,

$$\mathbf{B}^{p,n} = \text{diag}\left(\frac{1}{\Delta} + (r + m^u + \lambda)\right) - \mathbf{C}^p$$
$$\mathbf{b}^{p,n} = (c^u + m^u)\mathbf{1} + \lambda\bar{\mathbf{p}} + \frac{1}{\Delta}\mathbf{p}^n$$

- In the standard form of a LCP:

$$\mathbf{p}^{n+1\prime}(\mathbf{B}^{p,n}\mathbf{p}^{n+1} + (-\mathbf{b}^{p,n})) = 0$$
$$\mathbf{p}^{n+1} \geq 0$$
$$\mathbf{B}^{p,n}\mathbf{p}^{n+1} + (-\mathbf{b}^{p,n}) \geq 0$$

Boundary Conditions

- A firm entering default cannot resume operations or accumulate more debt:

$$f = f_D, \forall t \Rightarrow \begin{bmatrix} \xi_D^p & \beta_D^p & \zeta_D^p \end{bmatrix} = 0$$

where, f_D corresponds to the region of the state space where the firm is defaulted.

6.13.4. Solution for Joint Equity and Short-Term Debt Value Function and Investment Policy at No-Debt Boundary

- The HJB in the continuation region for the no-debt, no-trade equilibrium is:

$$0 = \max_g \left\{ -(r - g + \delta)\, j + A - \theta(A - c^s\alpha) - \Phi(g) - c^s\alpha + \alpha(g - \delta) \right\}$$

- FOC with respect to investment policy:

$$\Phi'(g) = j + \alpha$$
$$\Rightarrow j = \Phi'(g) - \alpha$$

- Joint equity and short-term debt value function for a firm that never takes on debt (and so does not default):

$$
\begin{aligned}
j &= \mathbb{E}\left[\int_0^\infty \exp(-(r-g^*+\delta)t)[\left[A-\theta(A-c^s\alpha)-\Phi(g^*)-c^s\alpha+\alpha\left(g^*-\delta\right)\right]dt+\alpha\sigma dZ_t]\right] \\
&= \int_0^\infty \exp(-(r-g^*+\delta)t)\left[A-\theta(A-c^s\alpha)-\Phi(g^*)-c^s\alpha+\alpha\left(g^*-\delta\right)\right]dt \\
&= \frac{A-\theta(A-c^s\alpha)-\Phi(g^*)-c^s\alpha+\alpha\left(g^*-\delta\right)}{r-g^*+\delta}
\end{aligned}
$$

- Combining these two equations, we have:

$$
r+\delta = g^* + \frac{1}{\Phi'(g^*)-\alpha}[A-\theta(A-c^s\alpha)-\Phi(g^*)-c^s\alpha+\alpha\left(g^*-\delta\right)]
$$

- Since $\Phi'(g^*)-\alpha=\gamma g^*-\alpha$ is strictly increasing in g^* and $g^*>\alpha/\gamma$, RHS is increasing in g^* if:

$$
\begin{gathered}
r+\delta > g^* > \alpha/\gamma \Rightarrow \Phi(r+\delta) > \Psi > \Phi(\alpha/\gamma) A-\theta(A-c^s\alpha)-\Phi(g^*)-c^s\alpha+\alpha\left(g^*-\delta\right) > 0 \\
(1-\theta)A-((1-\theta)c^s+\delta)\alpha > \Phi(g^*)-\alpha g^*
\end{gathered}
$$

- Then, $g^* \in (\alpha/\gamma, r+\delta)$:

$$
\Phi(\alpha/\gamma)-\alpha(\alpha/\gamma) < (1-\theta)A-((1-\theta)c^s+\delta)\alpha < \Phi(r+\delta)-\alpha(r+\delta)
$$

- Rearrange terms and express the equation as a quadratic function in g^*:

$$
-\gamma g^{*2}+2\gamma(r+\delta)g^*+2\alpha[(1-\theta)c^s-r]-2(1-\theta)A=0
$$

- Then, we choose the smaller root so that $g^* \in (\alpha/\gamma, r+\delta)$:

$$
\begin{aligned}
g^* &= \frac{-2\gamma(r+\delta)+\sqrt{(2\gamma(r+\delta))^2-4(-\gamma)(2\alpha[(1-\theta)c^s-r]-2(1-\theta)A)}}{-2\gamma} \\
&= \frac{\gamma(r+\delta)-\sqrt{(\gamma(r+\delta))^2+2\gamma(\alpha[(1-\theta)c^s-r]-(1-\theta)A)}}{\gamma}
\end{aligned}
$$

6.13.5. No-Debt Boundary Optimal Investment in Crisis

- The HJB in the continuation region for the no-debt, no-trade equilibrium with crisis dynamics is:

$$0 = \max_g \left\{ -(r - g + \delta)\, j + \eta A - \theta(\eta A - c^s\alpha) - \Phi(g) - c^s\alpha + (p_s^* - 1)\alpha + p_s^*\alpha(g - \delta) + \lambda(\bar{j} - j) \right\}$$

where $\bar{j}$ is the pre-shock joint equity and short-term debt value.

- FOC with respect to investment policy:

$$\Phi'(g) = j + p_s^*\alpha$$
$$\Rightarrow j = \Phi'(g) - p_s^*\alpha = \gamma g - p_s^*\alpha$$

- The joint equity and short-term debt value function for a firm that never takes on debt (and so does not default):

$$j^* = \frac{\eta A - \theta(\eta A - c^s\alpha) - \Phi(g^*) - c^s\alpha + (p_s^* - 1)\alpha + p_s^*\alpha(g^* - \delta) + \lambda\bar{j}}{r - g^* + \delta + \lambda}$$

- Combining these two equations, we have:

$$r + \delta + \lambda = g^* + \frac{1}{\Phi'(g^*) - p_s^*\alpha}[\eta A - \theta(\eta A - c^s\alpha) - \Phi(g^*) - c^s\alpha + (p_s^* - 1)\alpha + p_s^*\alpha(g^* - \delta) + \lambda\bar{j}]$$

- As a quadratic function in g^*,

$$\begin{aligned}
r + \delta + \lambda &= g^* + \frac{1}{\gamma g^* - p_s^*\alpha}[\eta A - \theta(\eta A - c^s\alpha) - \frac{1}{2}\gamma g^{*2} - c^s\alpha + (p_s^* - 1)\alpha \\
&\quad + p_s^*\alpha(g^* - \delta) + \lambda\bar{j}] \\
&= \frac{1}{\gamma g^* - p_s^*\alpha}[\gamma g^{*2} - g^* p_s^*\alpha + \eta A - \theta(\eta A - c^s\alpha) - \frac{1}{2}\gamma g^{*2} - c^s\alpha \\
&\quad + (p_s^* - 1)\alpha + p_s^*\alpha(g^* - \delta) + \lambda\bar{j}] \\
&= \frac{\frac{1}{2}\gamma g^{*2} + \eta A - \theta(\eta A - c^s\alpha) - c^s\alpha + (p_s^* - 1)\alpha - p_s^*\alpha\delta + \lambda\bar{j}}{\gamma g^* - p_s^*\alpha} \\
(r + \delta + \lambda)(\gamma g^* - p_s^*\alpha) &= \frac{1}{2}\gamma g^{*2} + \eta A - \theta(\eta A - c^s\alpha) - c^s\alpha + (p_s^* - 1)\alpha - p_s^*\alpha\delta + \lambda\bar{j} \\
\Rightarrow 0 &= -\gamma g^{*2} + 2\gamma(r + \delta + \lambda)g^* - 2p_s^*\alpha(r + \lambda) - 2[\eta A - \theta(\eta A - c^s\alpha) - c^s\alpha
\end{aligned}$$

$$+ (p_s^* - 1)\alpha + \lambda \bar{j})]$$

$$0 = -\gamma g^{*2} + 2\gamma(r + \delta + \lambda)g^*$$

$$- 2\alpha[p_s^*(r + \lambda) - (1 - \theta)c^s + (p_s^* - 1)] - 2[(1 - \theta)\eta A + \lambda \bar{j})]$$

- Choose smaller root:

$$g^* = \frac{-2\gamma(r + \delta + \lambda) + \sqrt{\begin{array}{c}(2\gamma(r + \delta + \lambda))^2 \\ -4(-\gamma)(-2\alpha[p_s^*(r + \lambda) + (1 - \theta)c^s + (p_s^* - 1)] - 2[(1 - \theta)\eta A + \lambda \bar{j})])\end{array}}}{-2\gamma}$$

$$= \frac{\gamma(r + \delta + \lambda) - \sqrt{\begin{array}{c}(\gamma(r + \delta + \lambda))^2 \\ -2\gamma(\alpha[p_s^*(r + \lambda) + (1 - \theta)c^s + (p_s^* - 1)] + [(1 - \theta)\eta A + \lambda \bar{j})])\end{array}}}{\gamma}$$

6.13.6. Primal-Dual Interior-Point Method

- To solve extensions to the base model with policy constraints, I marry an interior-point method to solve the resultant nonlinear optimization problem with an upwind finite differences method to solve the HJB and a linear complementary method to solve for the optimal stopping time (i.e. default).
- Below is the formulation for the nonlinear convex optimization problem featuring a dividend restriction, which both policies must satisfy.
- Define:

$$h(b^u, g, \omega) \equiv -(r - g + \delta + \lambda) j + \pi(b^u, g) + \lambda \bar{j} + [b^u - (g - \delta + m^u)f] j_f + \frac{1}{2}\sigma^2 f^2 j_{ff}$$

$$B(b^u, g, \omega) \equiv -h(b^u, g, \omega) - \omega \log(-\pi(b^u, g))$$

  Note the change in sign on the objective $B(b^u, g, \omega)$. This is because I'm solving the maximization problem as minimization.

- Compute the gradient of the barrier function $B(b^u, g, \omega)$:

$$\nabla B(b^u, g, \omega) = -\nabla h(b^u, g, \omega) - \omega \frac{1}{\pi(b^u, g)} \nabla \pi(b^u, g)$$

$$\begin{bmatrix} \frac{\partial B}{\partial b^u} \\ \frac{\partial B}{\partial g} \end{bmatrix} = - \begin{bmatrix} \frac{\partial h}{\partial b^u} \\ \frac{\partial h}{\partial g} \end{bmatrix} - \omega \frac{1}{\pi(b^u, g)} \begin{bmatrix} \frac{\partial \pi}{\partial b^u} \\ \frac{\partial \pi}{\partial g} \end{bmatrix}$$

$$= -\begin{bmatrix} \pi_{b^u} + j_f \\ j + \pi_g - f j_f \end{bmatrix} - \omega \frac{1}{\pi(b^u, g)} \begin{bmatrix} p \\ -\Phi'(g) + p_s^* \alpha \end{bmatrix}$$

$$= -\begin{bmatrix} p + j_f \\ j - \Phi'(g) + p_s^* \alpha - f j_f \end{bmatrix} - \omega \frac{1}{\pi(b^u, g)} \begin{bmatrix} p \\ -\Phi'(g) + p_s^* \alpha \end{bmatrix}$$

$$= -\begin{bmatrix} p + j_f \\ j - g\gamma + p_s^* \alpha - f j_f \end{bmatrix} - \omega \frac{1}{\pi(b^u, g)} \begin{bmatrix} p \\ -g\gamma + p_s^* \alpha \end{bmatrix}$$

6.14. Numerical Results for Model with Short-Term Debt

- Associated with Section 4.

6.14.1. Initial Distribution

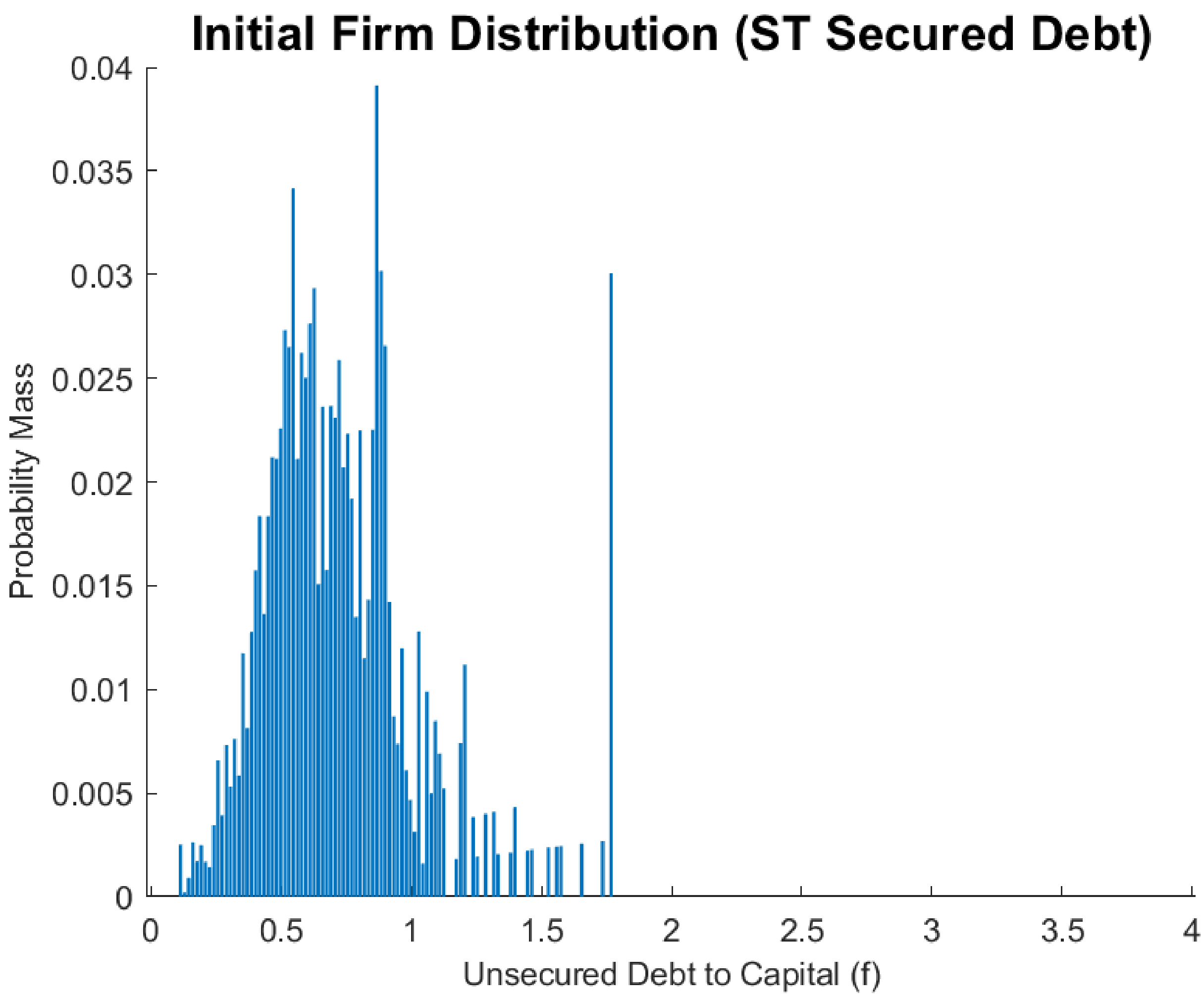


Figure A1. Initial Distribution of Firms

The figure plots the initial distribution of unsecured debt to capital. It is proxied by the variable `debt_capital` from the Financial Ratios Suite by Wharton WRDS based on Compustat data for rated firms. Ratings information is obtained from the Mergent FISD database. Data is winsorized at the 1% and 99% levels. The numerator of `debt_capital` is computed as the sum of accounts payable (`ap`), total debt in current liabilities (`dlc`), and total long-term debt (`dltt`). The denominator is computed as the sum of debt, and the sum of total common equity (`ceq`) and preferred stock (`pstkrv`, `pstkl`, `pstk`). Results are robust to using the variable `debt_assets`, which is computed as the ratio of total liabilities (`ltq`) to total assets (`at`).

6.14.2. Parameters

	Baseline	Crisis No. Int.	Crisis Unsec. Int.	Crisis Sec. Int.	Crisis Div. Rest.
A	0.24	$\eta \times 0.24$	$\eta \times 0.24$	$\eta \times 0.24$	$\eta \times 0.24$
$r^{(e)}$	0.05	0.05	0.05	0.05	0.05
$r^{(d)}$	$r^{(e)}$	$r^{(e)}$	$r^{(e)} - 0.02$	$r^{(e)}$	$r^{(e)} - 0.02$
δ	0.1	0.1	0.1	0.1	0.1
m^u	0.1	0.1	0.1	0.1	0.1
θ	0.35	0.35	0.35	0.35	0.35
α	0.20	0.20	0.20	0.20	0.20
σ	0.31	0.31	0.31	0.31	0.31
γ	16	16	16	16	16
η	1	0.95	0.95	0.95	0.95
λ	0	0.50	0.50	0.50	0.50
p_s^*	1	1	1	1.02	1

6.15. Dividend and Debt Repurchase Restrictions with Unsecured Debt Intervention

Figure A2 shows that the dividend restriction binds over the same region for both economies with and without unsecured debt repurchase restrictions. Unsurprisingly, payouts are far lower in the case with dividend restrictions.

Figure A3 shows firms unsecured debt issuance policies. Recall that the economies featuring dividend restrictions also benefit from unsecured debt intervention. In spite of this, firms in these economies do not choose to issue unsecured debt while the dividend restriction binds. Unable to make payouts, firms either increase investment, repurchase unsecured debt, or do both.

Figure A4 shows that dividend restrictions result in lower joint valuations for equity and short-term debt, particularly for the region of the state space where the dividend constraint binds. In fact, the difference between unconstrained and constrained equity prices increases as distance to the dividend restriction boundary increases. This is consistent with the monotonic pattern implied by the difference in dividends shown in Figure A2. The combination of lower equity prices, as well as non-positive unsecured debt issuance, suggest that firms would not voluntarily participate in credit programs featuring dividend and debt repurchase restrictions, which can explain the low uptake of the MSLP (which had less than a 3% utilization rate).

As intended, dividend restrictions sharply boost investment rates, as shown in Figure A5. In particular, a restriction on unsecured debt repurchases leads to sizeable

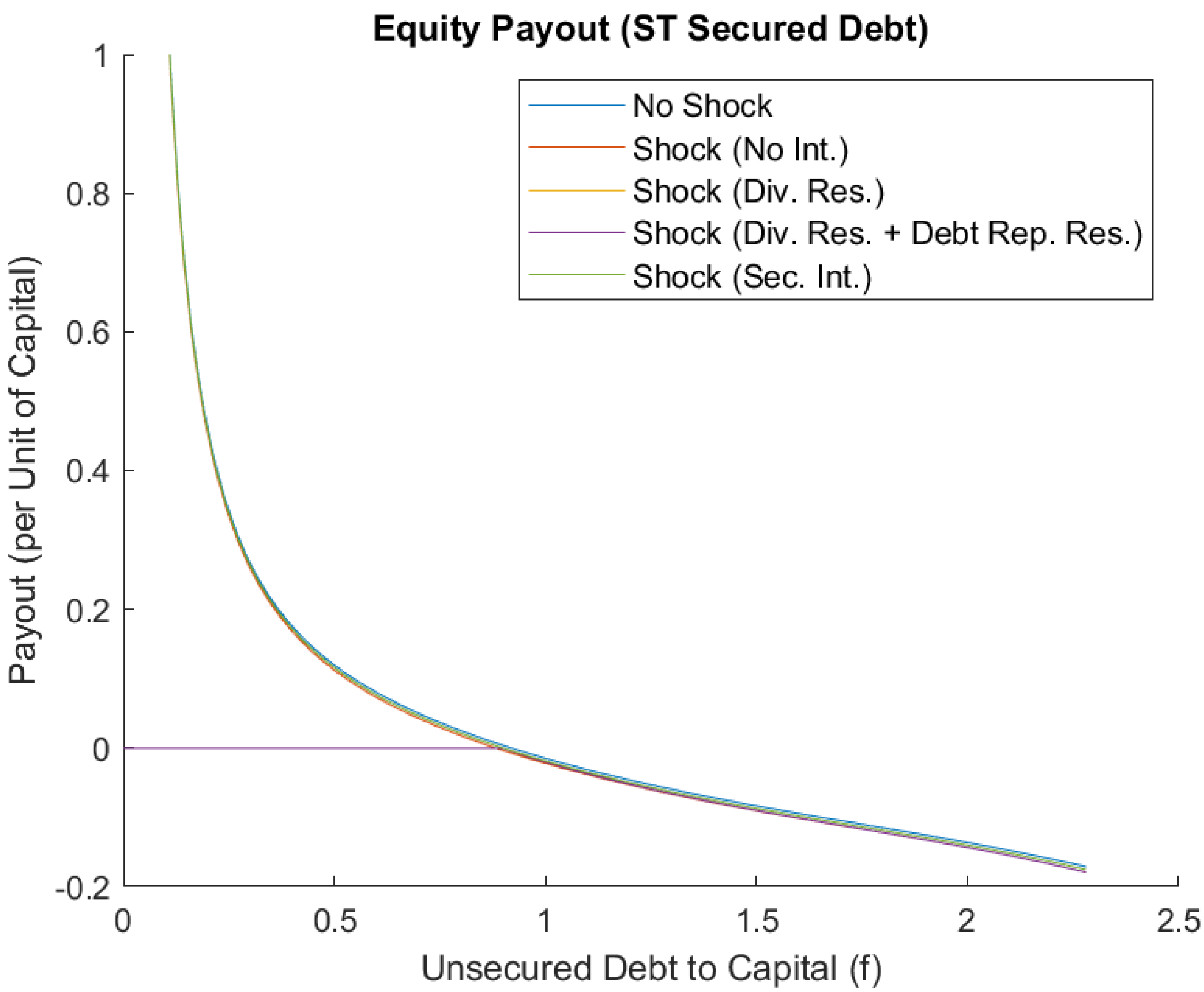


Figure A2. Dividend Restriction Binds for Large Portion of State Space

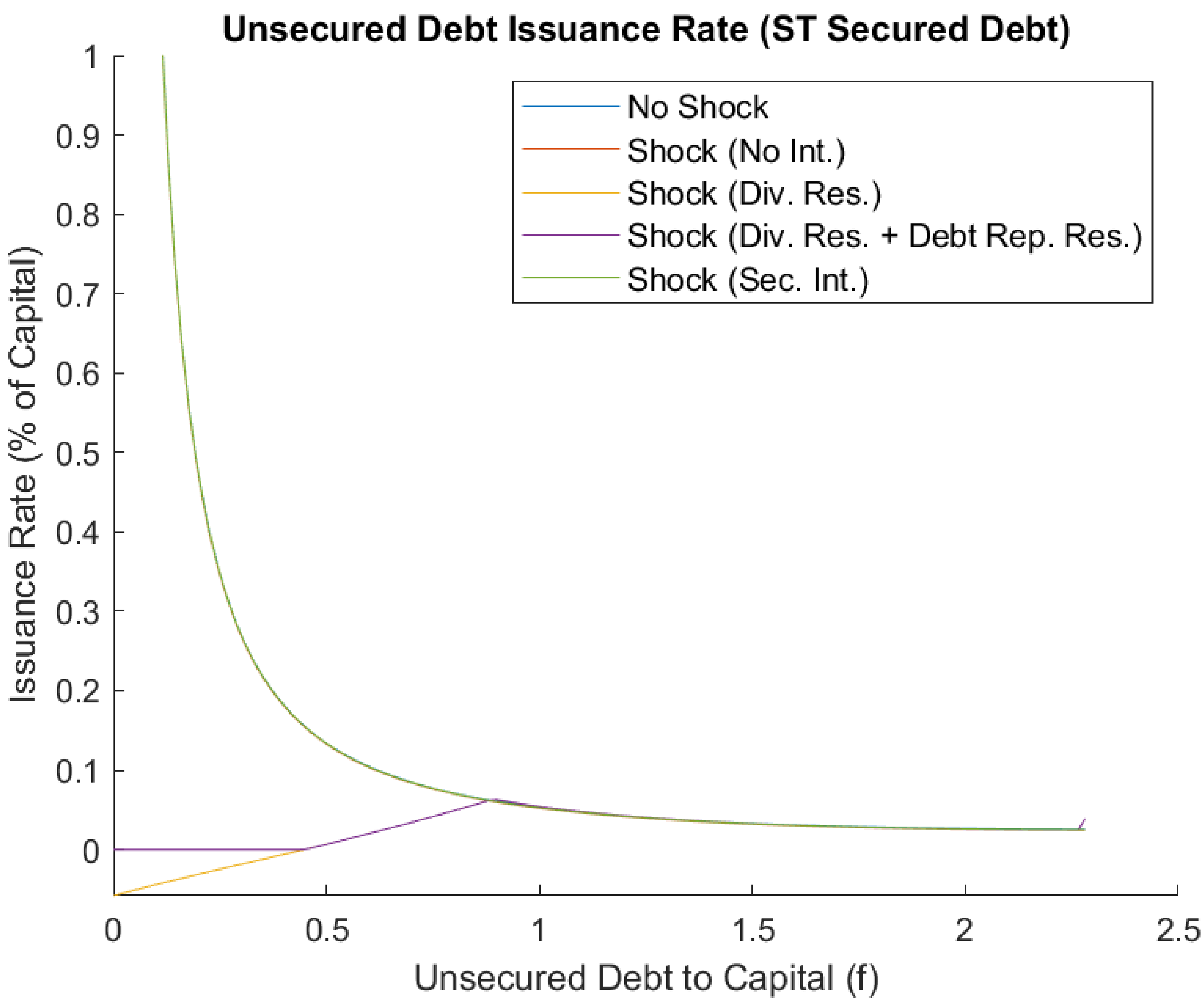
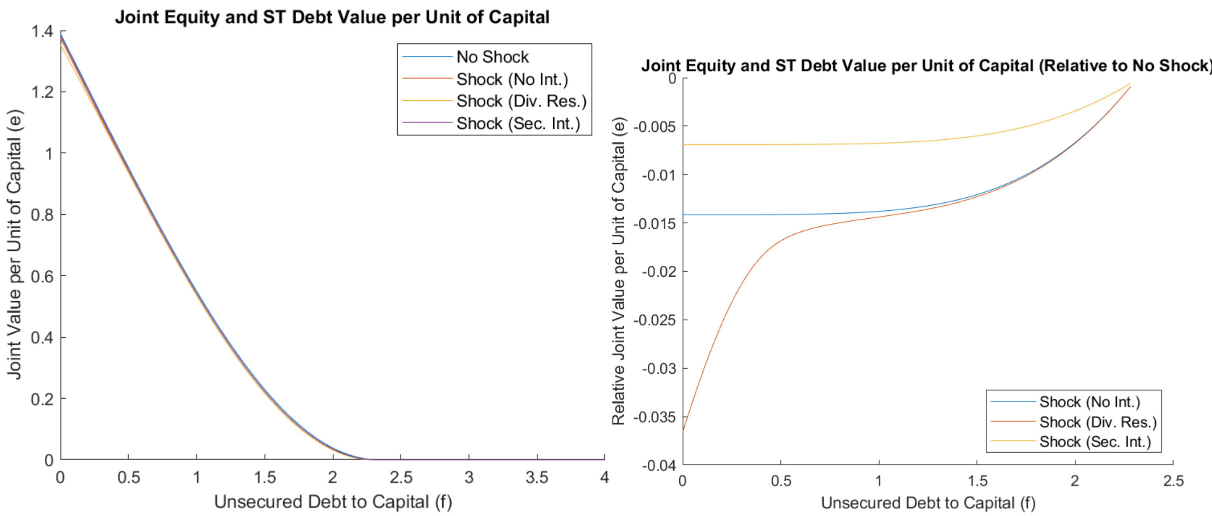


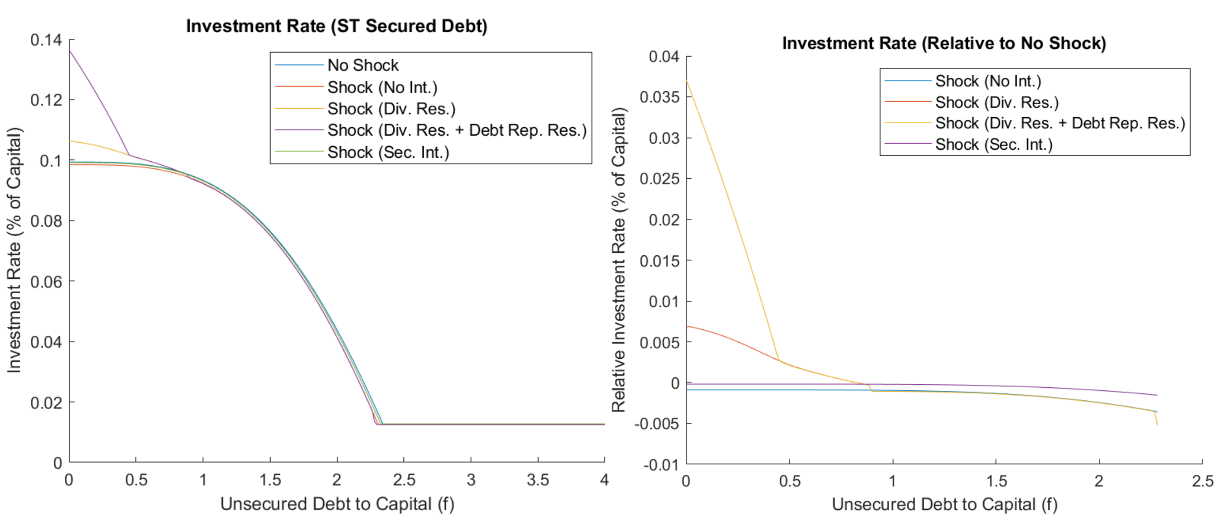
Figure A3. Debt Repurchase Motive With Dividend Restriction



A. Levels

B. Relative to No Shock

Figure A4. Dividend Restriction Decreases Equity Value



A. Levels

B. Relative to No Shock

Figure A5. Dividend Restriction Sharply Increases Investment Rates increase in investment. This reinforces the debt repurchase motive for firms, as seen in Figure A3.

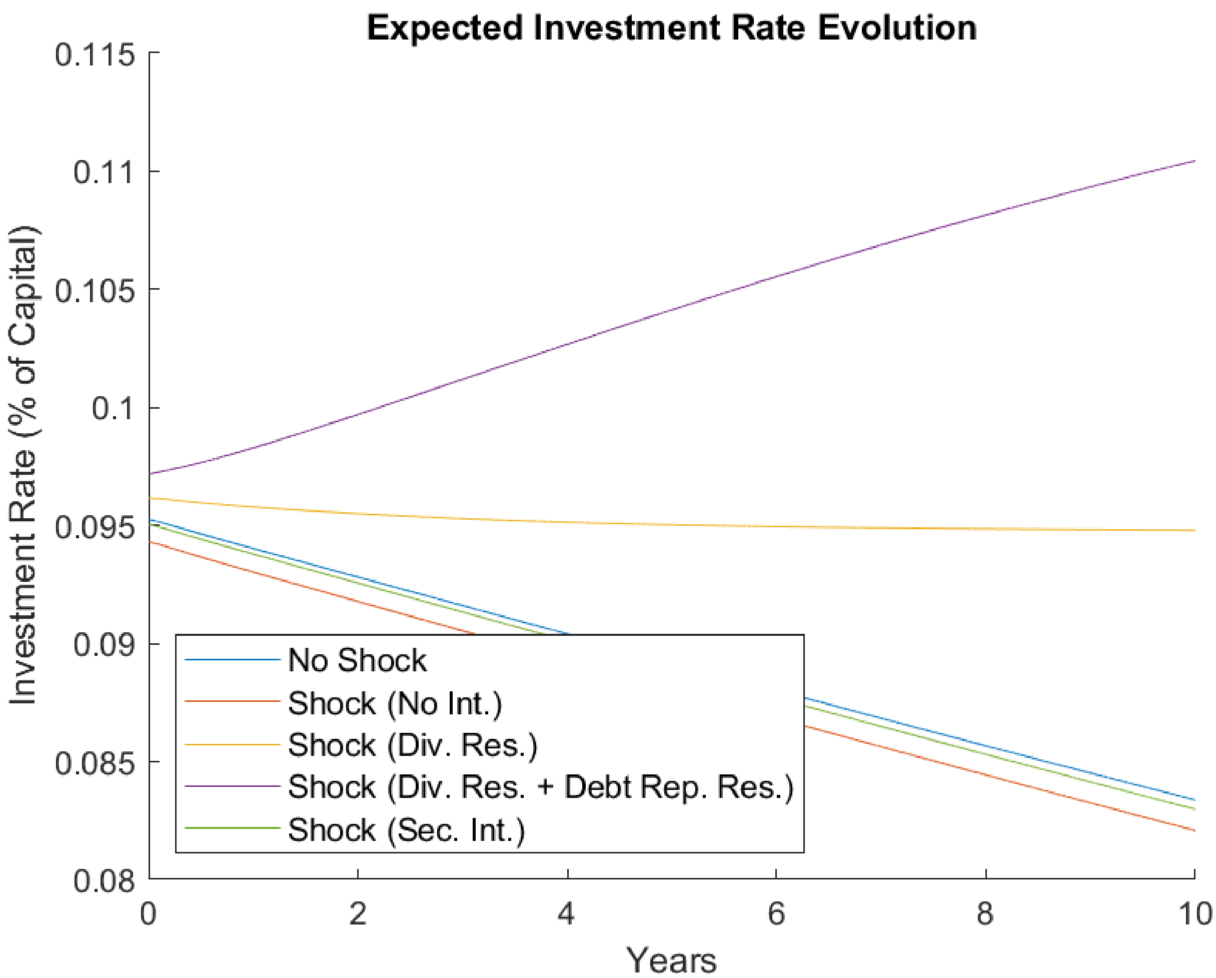


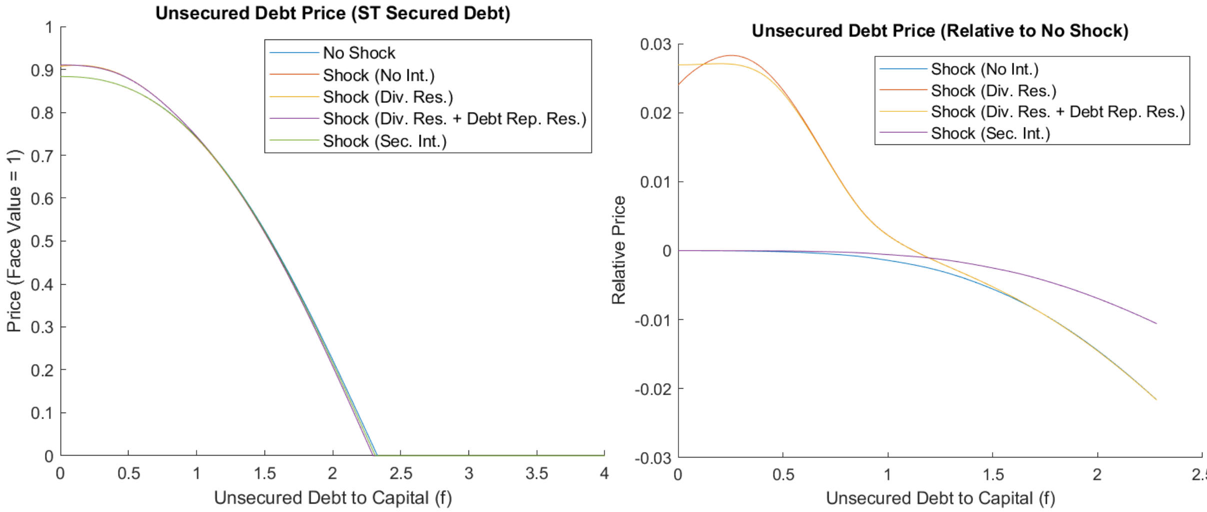
Figure A6. Expected Evolution of Investment Rates Also Far Higher

Echoing the differences in investment policy, the expected evolution of average investment rates is higher when dividend restrictions are in place, more so when the firm cannot repurchase unsecured debt. This is shown in Figure A6.

While equity prices are shown to fall in Figure A4, Figure A7 shows that unsecured debt price rises with dividend restrictions. With positive investment rates and no unsecured debt issuance, or even repurchases, the firm deleverages, reducing default risk and improving debt prices.

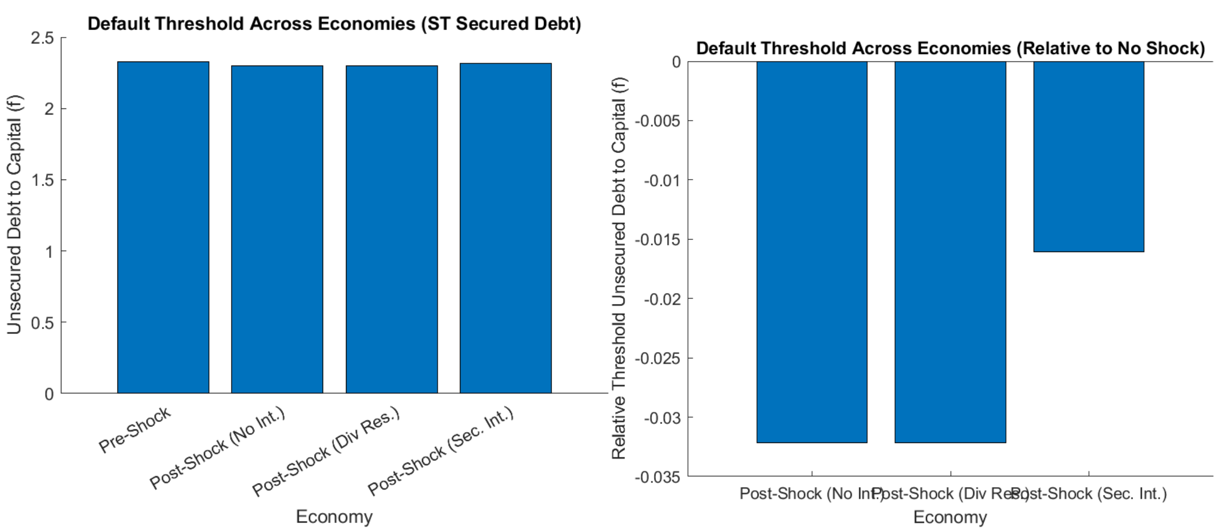
Figure A8 shows that dividend restrictions has no impact on the firm's default threshold relative to no intervention.

Figure A9 shows that the evolution of expected default rates are eventually lower



A. Levels B. Relative to No Shock

Figure A7. Dividend Restriction Increases Unsecured Debt Price



A. Levels B. Relative to No Shock

Figure A8. Dividend Restriction has No Impact on Default Threshold

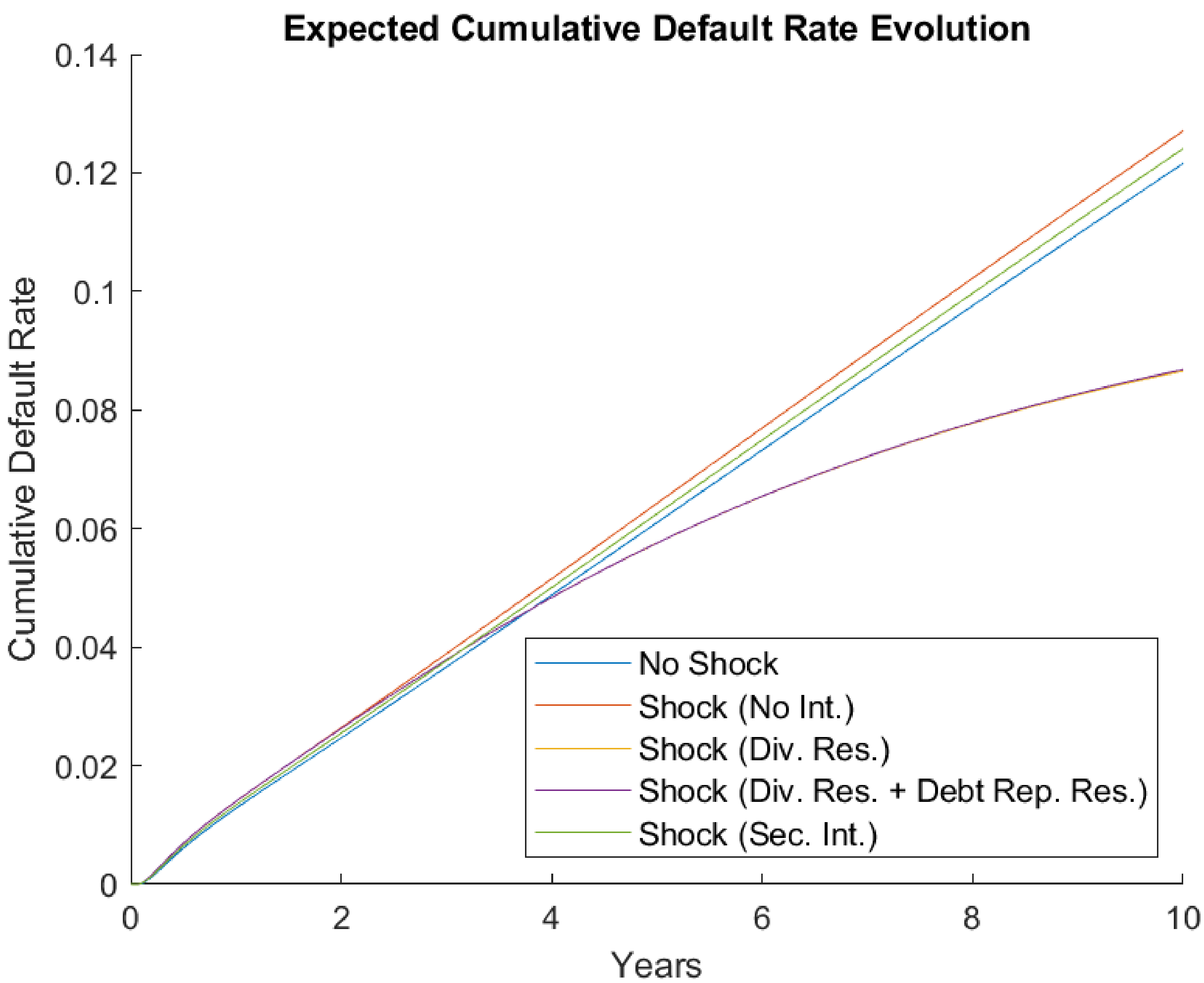


Figure A9. Dividend Restriction Sharply Reduces Long-Run Defaults with dividend restrictions. This is because the restriction forces the firm to grow larger (and hence, hold a smaller proportion of debt to assets). Additionally, when there are no restrictions on unsecured debt repurchases in place, retiring debt results in lower leverage and lowers default risk.

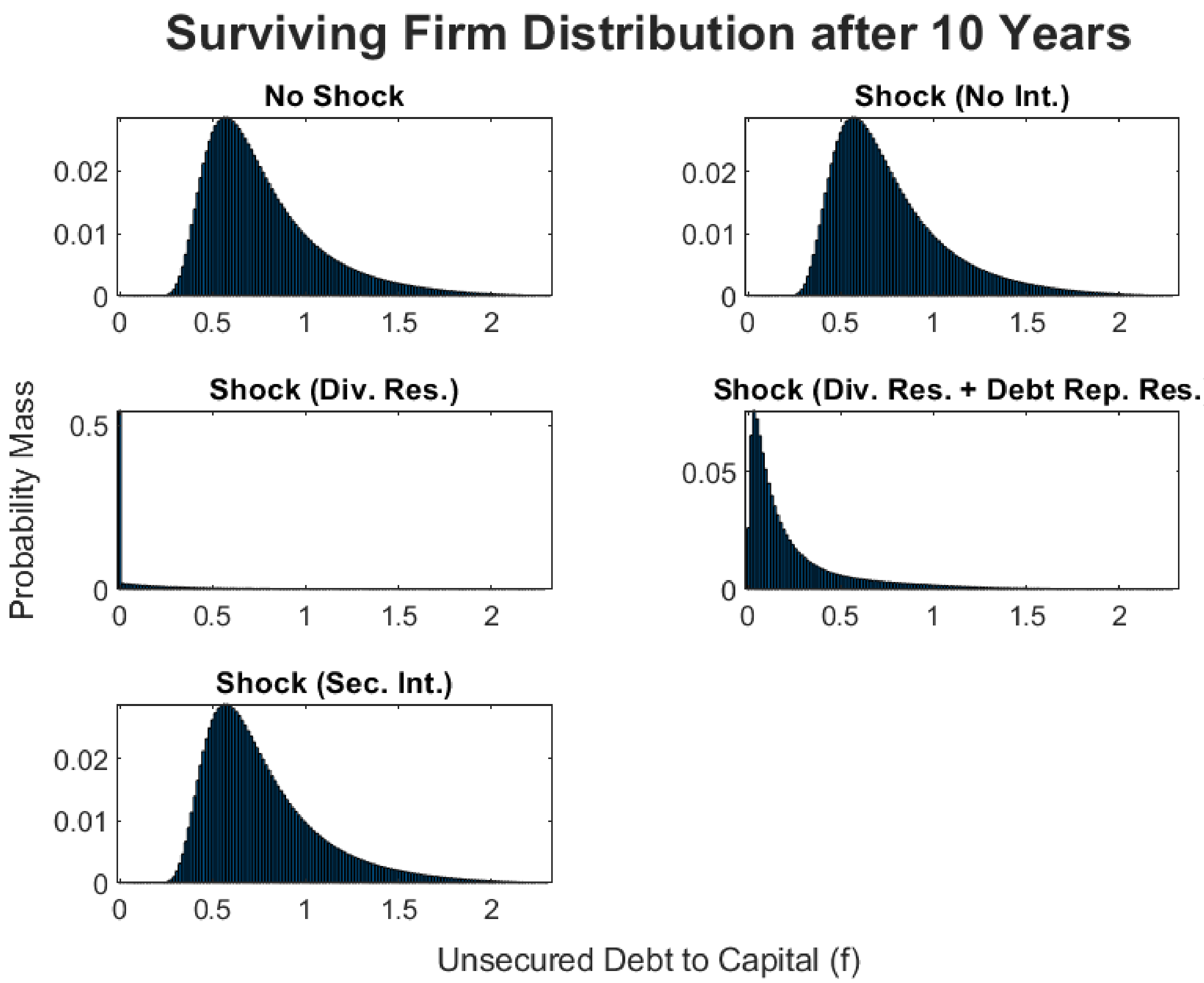


Figure A10. Surviving Firms Have Far Lower Leverage With Dividend Restriction; Even Less Without Debt Repurchase Restriction

Figure A11 shows a stark contrast in the distributions of surviving firms in the economies with and without dividend restrictions. Dividend restrictions cause firms to move away from the default boundary as a result of higher investment. This force is compounded when the firm can repurchase unsecured debt.

Figure A11 reinforces these results and gives a sense of how lower of a leverage ratio surviving firms have with dividend restrictions.

While it may be surprising that average expected equity value increases with dividend restrictions, as shown in Figure A12, this is because more firms are moving

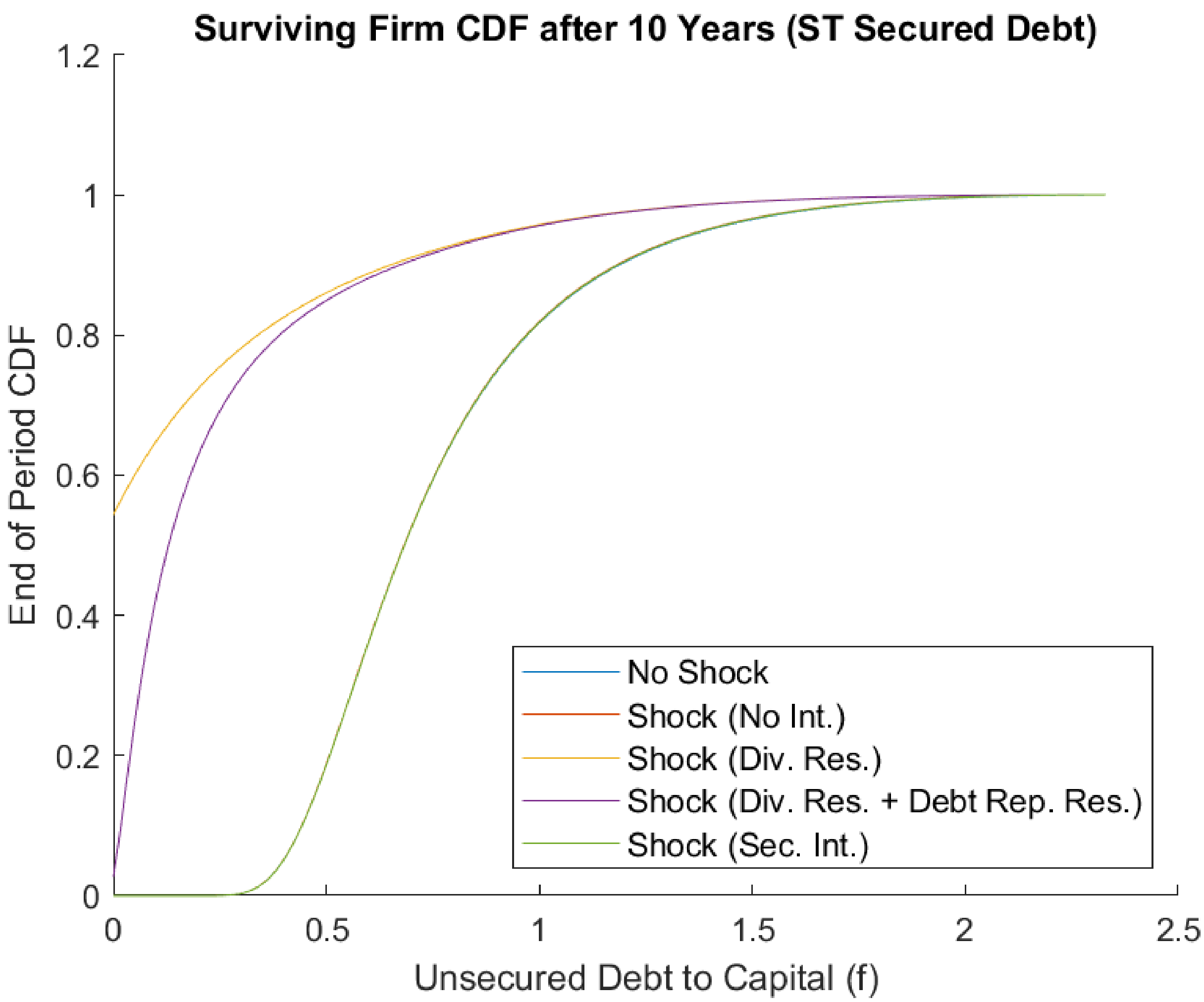


Figure A11. Surviving Firms Have Far Lower Leverage With Dividend Restriction; Even Less Without Debt Repurchase Restriction

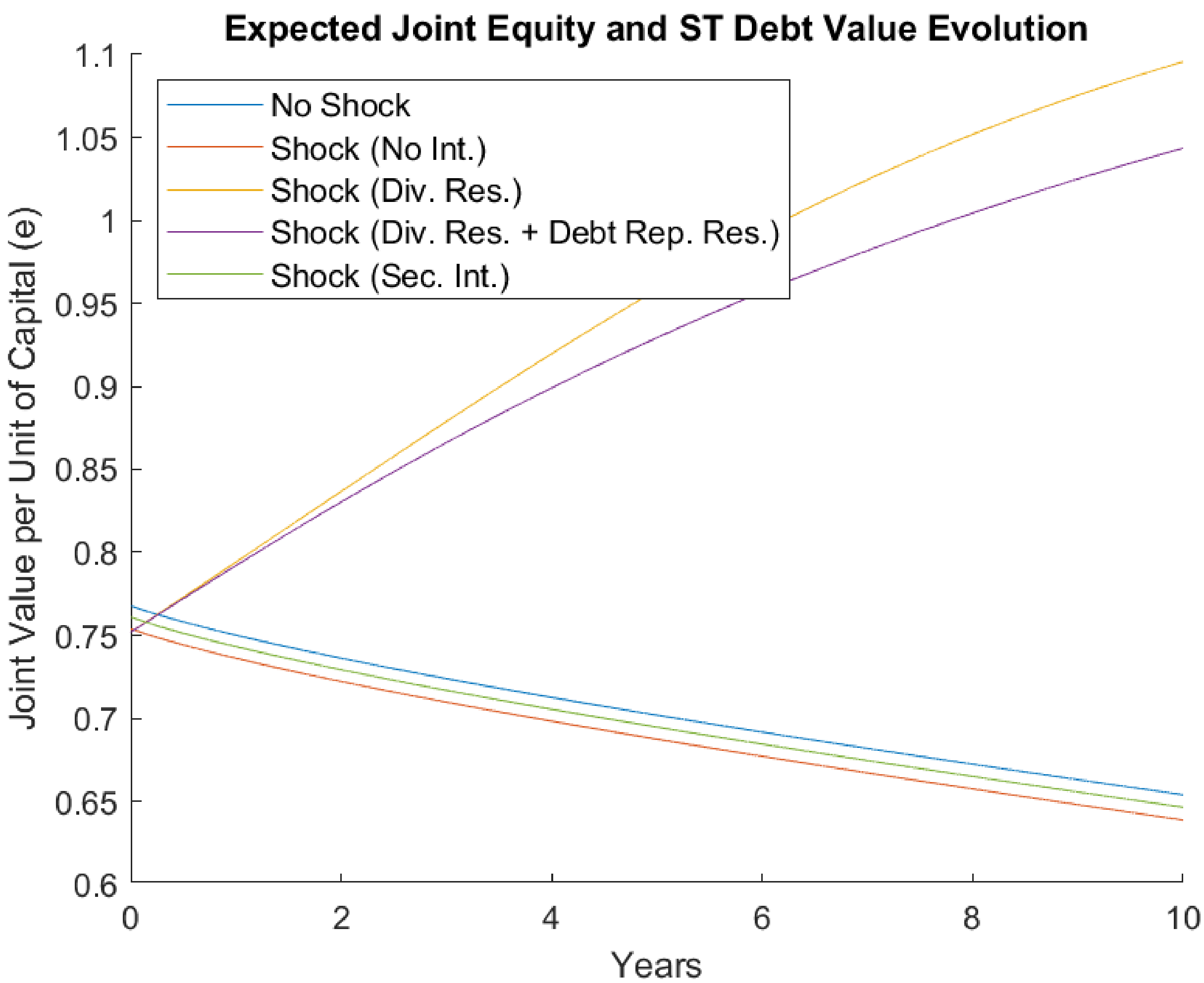


Figure A12. Expected Average Equity Values Increase Over Time With Restrictions away faster from the boundary than those which are defaulting. This force is strong enough to counteract the relatively lower equity prices from the dividend restriction binding more tightly.